UNDER PEER REVIEW

Minimal Surfaces on Time Scales

Contents

Preface

Minimal surfaces are among the most natural objects in Differential Geometry, and have been studied for the past 250 years ever since the pioneering work of Lagrange. The subject is characterized by a profound beauty, but perhaps even more remarkably, minimal surfaces (or minimal submanifolds) have encountered striking applications in other fields, like three-dimensional topology, mathematical physics, conformal geometry, among others. Even though it has been the subject of intense activity, many basic open problems still remain.

The time scales theory projected by Stefan Hilger in 1988 unifies the study of continuous and discrete analysis. Since then, it has been used intensively by many researchers working in different areas of mathematics. The main goal of this book is to find suitable time scale analog of minimal surfaces. This class of surfaces is considered in the context of the dynamic geometry on time scales.

The book contains four chapters. In Chapter 1 we introduce a complex integral with a real variable and some of its properties are derived. Countour integrals are defined and some of their properties are proved. Chapter 2 is devoted to the local theory of minimal surfaces on time scales. Parametric surfaces, nonparametric surfaces, first and second fundamental forms of a surface, surfaces that minimize area are introduced and developed. A time scale analog of the Bernsten theorem is formulated and proved. In Chapter 3 we introduce the global theory of minimal surfaces on time scales. They are defined σ_1-n-manifolds and some of their properties are deduced. The main equation of a minimal surface on time scales is derived. The Gauss map is defined and some of its basic properties are deduced. The Gauss curvature and the total curvature are introduced and investigated. Chapter 4 is devoted to a variational approach for studying minimal surfaces on time scales.

The aim of this book is to present a clear and well-organized treatment of the concept behind the development of mathematics and solution techniques. The text material of this book is presented in highly readable, mathematically solid format.

Chapter 1
Complex Integration on Time Scales

1.1 Complex Integral with a Real Variable

Let $\mathbb{T}$ be a time scale with forward jump operator and delta differentiation operator σ and Δ, respectively. Let also, $[a,b] \subset \mathbb{T}$. Consider a complex-valued function of a real variable t

$$f(t) = u(t) + iv(t), \quad t \in [a,b],$$

where $u, v \in \mathscr{C}_{rd}([a,b])$.

Definition 1.1. The integral of f from $t = a$ to $t = b$ is defined as follows

$$\int_a^b f(t)\Delta t = \int_a^b u(t)\Delta t + i\int_a^b v(t)\Delta t.$$

Remark 1.1. Since $u, v \in \mathscr{C}_{rd}([a,b])$, we have that both integrals

$$\int_a^b u(t)\Delta t \quad \text{and} \quad \int_a^b v(t)\Delta t$$

exist and then the integral $\int_a^b f(t)\Delta t$ is well defined.

Example 1.1. Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Consider

$$f(t) = (t^2+1)it, \quad t \in [1,8].$$

Here

$$\sigma(t) = 2t,$$

$$u(t) = t^2 + 1,$$

$$v(t) = t, \quad t \in [1, 8].$$

Let

$$h(t) = \frac{1}{7}t^3 + t,$$

$$g(t) = \frac{1}{3}t^2, \quad t \in [1, 8].$$

We have

$$\begin{aligned} h^{\Delta}(t) &= \frac{1}{7}((\sigma(t))^2 + t\sigma(t) + t^2) \\ &= \frac{1}{7}(4t^2 + 2t^2 + t^2) \\ &= t^2, \quad t \in [1, 8], \end{aligned}$$

and

$$\begin{aligned} g^{\Delta}(t) &= \frac{1}{3}(\sigma(t) + t) \\ &= \frac{1}{3}(2t + t) \\ &= t, \quad t \in [1, 8]. \end{aligned}$$

Hence,

$$\begin{aligned} \int_1^8 f(t)\Delta t &= \int_1^8 u(t)\Delta t + i\int_1^8 v(t)\Delta t \\ &= \int_1^8 (t^2 + 1)\Delta t + i\int_1^8 t\Delta t \\ &= \int_1^8 h^{\Delta}(t)\Delta t + i\int_1^8 g^{\Delta}(t)\Delta t \\ &= h(t)\Big|_{t=1}^{t=8} + ig(t)\Big|_{t=1}^{t=8} \end{aligned}$$

$$= \left(\frac{1}{7}t^3 + t\right)\Big|_{t=1}^{t=8} + it\Big|_{t=1}^{t=8}$$

$$= \left(\frac{512}{7} + 8 - \frac{1}{7} - 1\right) + i(8-1)$$

$$= \left(\frac{511}{7} + 7\right) + 7i$$

$$= \frac{560}{7} + 8i.$$

The basic properties of a complex integral with a real variable of integration are as follows.

1.

$$\operatorname{Re}\int_a^b f(t)\Delta t = \int_a^b u(t)\Delta t,$$

$$\operatorname{Im}\int_a^b f(t)\Delta t = \int_a^b v(t)\Delta t.$$

Proof. We have

$$\int_a^b f(t)\Delta t = \int_a^b u(t)\Delta t + i\int_a^b v(t)\Delta t,$$

whereupon we get the desired result.

2. Let

$$f_j(t) = u_j(t) + iv_j(t), \quad t \in [a,b],$$

$$\gamma_j = a_j + ib_j, \quad j \in \{1,2\},$$

where $u_j, v_j \in \mathscr{C}_{rd}([a,b])$, $j \in \{1,2\}$, are real-valued functions and $a_j, b_j \in \mathbb{R}$, $j \in \{1,2\}$. Then

$$\int_a^b (\gamma_1 f_1(t) + \gamma_2 f_2(t))\Delta t = \gamma_1\int_a^b f_1(t)\Delta t + \gamma_2\int_a^b f_2(t)\Delta t.$$

Proof. Note that

$$\operatorname{Re}(\gamma_j f_j(t)) = a_j u_j(t) - b_j v_j(t),$$

$$\operatorname{Im}(\gamma_j f_j(t)) = a_j v_j(t) + b_j u_j(t), \quad j \in \{1,2\}, \quad t \in [a,b],$$

and

$$\gamma_1 f_1(t) + \gamma_2 f_2(t) = (a_1 u_1(t) + a_2 v_2(t) - b_1 v_1(t) - b_2 v_2(t))$$

$$+ i(a_1 v_1(t) + b_1 u_1(t) + a_2 v_2(t) + b_2 u_2(t)), \quad t \in [a,b].$$

Hence,

$$\int_a^b (\gamma_1 f_1(t) + \gamma_2 f_2(t))\Delta t = \int_a^b (a_1 u_1(t) + a_2 u_2(t) - b_1 v_1(t) - b_2 v_2(t))\Delta t$$

$$+ i \int_a^b (a_1 v_1(t) + b_1 u_1(t) + a_2 v_2(t) + b_2 u_2(t))\Delta t$$

$$= a_1 \left(\int_a^b u_1(t)\Delta t + i \int_a^b v_1(t)\Delta t \right) + b_1 \left(- \int_a^b v_1(t)\Delta t + i \int_a^b u_1(t)\Delta t \right)$$

$$+ a_2 \left(\int_a^b u_2(t)\Delta t + i \int_a^b v_2(t)\Delta t \right) + b_2 \left(- \int_a^b v_2(t)\Delta t + i \int_a^b u_2(t)\Delta t \right)$$

$$= a_1 \int_a^b f_1(t)\Delta t + i b_1 \left(\int_a^b u_1(t)\Delta t + i \int_a^b v_1(t)\Delta t \right)$$

$$+ a_2 \int_a^b f_2(t)\Delta t + i b_1 \left(\int_a^b u_2(t)\Delta t + i \int_a^b v_2(t)\Delta t \right)$$

$$= a_1 \int_a^b f_1(t)\Delta t + i b_1 \int_a^b f_1(t)\Delta t + a_2 \int_a^b f_2(t)\Delta t + i b_2 \int_a^b f_2(t)\Delta t$$

$$= (a_1 + i b_1) \int_a^b f_1(t)\Delta t + (a_2 + i b_2) \int_a^b f_2(t)\Delta t$$

$$= \gamma_1 \int_a^b f_1(t)\Delta t + \gamma_2 \int_a^b f_2(t)\Delta t.$$

This completes the proof.

3.

$$\left|\int_a^b f(t)\Delta t\right| \le \int_a^b |f(t)|\Delta t.$$

Proof. Note that

$$\left|\int_a^b f(t)\Delta t\right| = e^{-i\phi}\int_a^b f(t)\Delta t,$$

where

$$\phi = \operatorname{Arg}\left(\int_a^b f(t)\Delta t\right).$$

Hence,

$$\begin{aligned}\left|\int_a^b f(t)\Delta t\right| &= \operatorname{Re}\left(\int_a^b e^{-i\phi} f(t)\Delta t\right)\\ &= \int_a^b \operatorname{Re}\left(e^{-i\phi} f(t)\right)\Delta t\\ &\le \int_a^b \left|e^{-i\phi} f(t)\right|\Delta t\\ &= \int_a^b |f(t)|\Delta t.\end{aligned}$$

This completes the proof.

1.2 Definition of a Countour Integral

Consider a curve C which is a set of points $z = (x, y)$ in the complex plane defined by

$$x = x(t),$$

$$y = y(t), \quad t \in [a, b],$$

where $x, y \in \mathscr{C}^1_{rd}([a, b])$. We can write

$$z(t) = x(t) + iy(t), \quad t \in [a, b].$$

Definition 1.2. A countour is defined as a curve consisting of a finite number of smooth curves joined end to end

Definition 1.3. A countour is said to be a simple connected contour if the initial and final values of $z(t)$ are the same and the curve does not cross itself.

Suppose that

$$f(z) = u(x,y) + iv(x,y),$$

$$dz = dx + idy.$$

Definition 1.4. The countour integral of $f(z)$ along the contour C is defined to be the integral

$$\int_C f(z)\Delta z = \int_a^b f(z(t))z^{\Delta}(t)\Delta t, \tag{1.1}$$

where

$$z^{\Delta}(t) = x^{\Delta}(t) + iy^{\Delta}(t), \quad t \in [a,b].$$

The contour integral (1.1) can be represented in the form

$$\int_C f(z)\Delta z = \int_a^b (u(x(t),y(t)) + iv(x(t),y(t)))(x^{\Delta}(t) + iy^{\Delta}(t))\Delta t$$

$$= \int_a^b (u(x(t),y(t))x^{\Delta}(t) - v(x(t),y(t))y^{\Delta}(t))\Delta t$$

$$+ \int_a^b (u(x(t),y(t))y^{\Delta}(t) + v(x(t),y(t))x^{\Delta}(t))\Delta t.$$

By the usual properties of real line integrals, it follows the following.

1. $\int_C f(z)\Delta z$ is independent of the parameterization of C.
2. $$\int_{-C} f(z)\Delta z = -\int_C f(z)\Delta z,$$
   where $-C$ is the opposite curve of C.
3. The integrals of $f(z)$ along a string of contours is equal to the sum of integrals of $f(z)$ along each of these contours.

Example 1.2. We will evaluate

$$\int_C |z|^2 \Delta z,$$

where

$$C: \begin{array}{l} x(t) = t, \\ y(t) = t+1, \quad t \in [1,9]. \end{array}$$

We have

$$\sigma(t) = 3t,$$

$$\begin{aligned} |z(t)|^2 &= (x(t))^2 + (y(t))^2 \\ &= t^2 + (t+1)^2 \\ &= t^2 + t^2 + 2t + 1 \\ &= 2t^2 + 2t + 1, \\ z(t) &= x(t) + iy(t) \\ &= t + i(t+1), \quad t \in [1,9], \end{aligned}$$

and

$$\begin{aligned} \Delta z(t) &= \Delta t + i\Delta t \\ &= (1+i)\Delta t, \quad t \in [1,9]. \end{aligned}$$

Let

$$g(t) = \frac{2}{13}t^3 + \frac{1}{2}t^2 + t, \quad t \in [1,9].$$

Then

$$\begin{aligned} g^{\Delta}(t) &= \frac{2}{13}((\sigma(t))^2 + t\sigma(t) + t^2) + \frac{1}{2}(\sigma(t) + t) \\ &= \frac{2}{13}(9t^2 + 3t^2 + t^2) + \frac{1}{2}(3t + t) + 1 \\ &= 2t^2 + 2t + 1, \quad t \in [1,9]. \end{aligned}$$

Therefore

$$\begin{aligned}\int_C |z|^2 \Delta z &= \int_1^9 (2t^2+2t+1)(1+i)\Delta t \\ &= (1+i)\int_1^9 g^{\Delta}(t)\Delta t \\ &= (1+i)g(t)\Big|_{t=1}^{t=9} \\ &= (1+i)\left(\frac{2}{13}t^3+\frac{1}{2}t^2+t\right)\Big|_{t=1}^{t=9} \\ &= (1+i)\left(\frac{2}{13}\cdot 729+\frac{81}{2}+9-\frac{2}{13}-\frac{1}{2}-1\right) \\ &= (1+i)\left(\frac{1458}{13}-\frac{2}{13}+40+8\right) \\ &= (1+i)\left(\frac{1456}{13}+48\right) \\ &= (1+i)(112+48) \\ &= 160(1+i).\end{aligned}$$

Theorem 1.1. *Let M be the upper bound of $|f(z)|$ along C and L be the arc;length of the curve C. Then*

$$\left|\int_C f(z)\Delta z\right| \le ML.$$

Proof. We have

$$\begin{aligned}\left|\int_C f(z)\Delta z\right| &= \left|\int_a^b f(z(t))z^{\Delta}(t)\Delta t\right| \\ &\le \int_a^b |f(z(t))||z^{\Delta}(t)|\Delta t \\ &\le M\int_a^b |z^{\Delta}(t)|\Delta t\end{aligned}$$

$$= ML.$$

This completes the proof.

Chapter 2
Local Theory

Suppose that $\mathbb{T}$, $\mathbb{T}_1$, $\mathbb{T}_2$, $\mathbb{T}_3$, $\mathbb{T}_{(1)}$, $\mathbb{T}_{(2)}$, $\mathbb{T}_{(3)}$ are time scales with forward jump operators and delta differentiation operators σ, σ_1, σ_2, σ_3, $\sigma_{(1)}$, $\sigma_{(2)}$, $\sigma_{(3)}$ and Δ, Δ_1, Δ_2, Δ_3, $\Delta_{(1)}$, $\Delta_{(2)}$ and $\Delta_{(3)}$, respectively. Let $I \subseteq \mathbb{T}$, $U, U_1, W_1 \subseteq \mathbb{T}_1 \times \mathbb{T}_2$, $\widetilde{U}$, $W_2 \subset \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$ and $V \subseteq \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{T}_3$.

2.1 Parametric Surfaces

We will denote by $x = (x_1, x_2, x_3)$ a point in $\mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$.

Definition 2.1. Any smooth map $x : U \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$,

$$x(u_1, u_2) = (x_1(u_1, u_2), x_2(u_1, u_2), x_3(u_1, u_2)), \quad (u_1, u_2) \in U,$$

is said to be a σ_1-parametric surface if x is σ_1-completely delta differentiable in U.

Denote the σ_1-Jacobian of the mapping x by

$$\begin{aligned} M &= (m_{ij}) \\ &= \begin{pmatrix} \dfrac{\partial x_1}{\Delta u_1} & \dfrac{\partial x_1}{\Delta u_2}^{\sigma_1} \\ \dfrac{\partial x_2}{\Delta u_1} & \dfrac{\partial x_2}{\Delta u_2}^{\sigma_1} \\ \dfrac{\partial x_3}{\Delta u_1} & \dfrac{\partial x_3}{\Delta u_2}^{\sigma_1} \end{pmatrix}. \end{aligned}$$

For any two vectors

$$y = (y_1, y_2, y_3) \quad \text{and} \quad z = (z_1, z_2, z_3)$$

of $\mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$, we denote the inner product by

$$\begin{aligned} y\cdot z &= y_1z_1+y_2z_2+y_3z_3\\ &= \sum_{j=1}^{3} y_jz_j \end{aligned}$$

and the cross product

$$y\times z = \left(\det\begin{pmatrix} y_2 & y_3\\ z_2 & z_3\end{pmatrix}, -\det\begin{pmatrix} y_1 & y_3\\ z_1 & z_3\end{pmatrix}, \det\begin{pmatrix} y_1 & y_2\\ z_1 & z_2\end{pmatrix}\right).$$

Now, we introduce the matrix

$$G=(g_{ij})=M^TM.$$

For G, we have the following representation

$$G=\begin{pmatrix} \frac{\partial x_1}{\Delta u_1} & \frac{\partial x_1}{\Delta u_2}^{\sigma_1}\\ \frac{\partial x_2}{\Delta u_1} & \frac{\partial x_2}{\Delta u_2}^{\sigma_1}\\ \frac{\partial x_3}{\Delta u_1} & \frac{\partial x_3}{\Delta u_2}^{\sigma_1}\end{pmatrix}^T \begin{pmatrix} \frac{\partial x_1}{\Delta u_1} & \frac{\partial x_1}{\Delta u_2}^{\sigma_1}\\ \frac{\partial x_2}{\Delta u_1} & \frac{\partial x_2}{\Delta u_2}^{\sigma_1}\\ \frac{\partial x_3}{\Delta u_1} & \frac{\partial x_3}{\Delta u_2}^{\sigma_1}\end{pmatrix}$$

$$=\begin{pmatrix} \frac{\partial x_1}{\Delta u_1} & \frac{\partial x_2}{\Delta u_1} & \frac{\partial x_3}{\Delta u_1}\\ \frac{\partial x_1}{\Delta u_2}^{\sigma_1} & \frac{\partial x_2}{\Delta u_2}^{\sigma_1} & \frac{\partial x_3}{\Delta u_2}^{\sigma_1}\end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\Delta u_1} & \frac{\partial x_1}{\Delta u_2}^{\sigma_1}\\ \frac{\partial x_2}{\Delta u_1} & \frac{\partial x_2}{\Delta u_2}^{\sigma_1}\\ \frac{\partial x_3}{\Delta u_1} & \frac{\partial x_3}{\Delta u_2}^{\sigma_1}\end{pmatrix}$$

$$=\begin{pmatrix} \left(\frac{\partial x_1}{\Delta u_1}\right)^2+\left(\frac{\partial x_2}{\Delta u_1}\right)^2+\left(\frac{\partial x_3}{\Delta u_1}\right)^2 & \frac{\partial x_1}{\Delta u_1}\frac{\partial x_1}{\Delta u_2}^{\sigma_1}+\frac{\partial x_2}{\Delta u_1}\frac{\partial x_2}{\Delta u_2}^{\sigma_1}+\frac{\partial x_3}{\Delta u_1}\frac{\partial x_3}{\Delta u_2}^{\sigma_1}\\ \frac{\partial x_1}{\Delta u_1}\frac{\partial x_1}{\Delta u_2}^{\sigma_1}+\frac{\partial x_2}{\Delta u_1}\frac{\partial x_2}{\Delta u_2}^{\sigma_1}+\frac{\partial x_3}{\Delta u_1}\frac{\partial x_3}{\Delta u_2}^{\sigma_1} & \left(\frac{\partial x_1}{\Delta u_2}^{\sigma_1}\right)^2+\left(\frac{\partial x_2}{\Delta u_2}^{\sigma_1}\right)^2+\left(\frac{\partial x_3}{\Delta u_2}^{\sigma_1}\right)^2\end{pmatrix}.$$

For the cross product of the vectors $\frac{\partial x}{\Delta u_1}$ and $\frac{\partial x}{\Delta u_2}^{\sigma_1}$, we have the representation

$$\frac{\partial x}{\Delta u_1}\times\frac{\partial x}{\Delta u_2}^{\sigma_1} = \left(\frac{\partial x_1}{\Delta u_1},\frac{\partial x_2}{\Delta u_1},\frac{\partial x_3}{\Delta u_1}\right)\times\left(\frac{\partial x_1}{\Delta u_2}^{\sigma_1},\frac{\partial x_2}{\Delta u_2}^{\sigma_1},\frac{\partial x_3}{\Delta u_2}^{\sigma_1}\right)$$

$$=\left(\frac{\partial x_2}{\Delta u_1}\frac{\partial x_3}{\Delta u_2}^{\sigma_1}-\frac{\partial x_3}{\Delta u_1}\frac{\partial x_2}{\Delta u_2}^{\sigma_1}, \frac{\partial x_1}{\Delta u_2}^{\sigma_1}\frac{\partial x_3}{\Delta u_1}-\frac{\partial x_1}{\Delta u_1}\frac{\partial x_3}{\Delta u_2}^{\sigma_1}, \frac{\partial x_1}{\Delta u_1}\frac{\partial x_2}{\Delta u_2}^{\sigma_1}-\frac{\partial x_1}{\Delta u_2}^{\sigma_1}\frac{\partial x_2}{\Delta u_1}\right).$$

We have the following σ_1-Lagrange identity.

Theorem 2.1. *Let $x : U \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$ be σ_1-completely delta differentiable. Then the σ_1-Lagrange identity holds*

$$\det G = \left| \frac{\partial x}{\Delta u_1} \times \frac{\partial x}{\Delta u_2}^{\sigma_1} \right|^2.$$

Proof. By the representation of the matrix G, we find

$$\begin{aligned}
\det G =& \left(\left(\frac{\partial x_1}{\Delta u_1} \right)^2 + \left(\frac{\partial x_2}{\Delta u_1} \right)^2 + \left(\frac{\partial x_3}{\Delta u_1} \right)^2 \right) \left(\left(\frac{\partial x_1}{\Delta u_2}^{\sigma_1} \right)^2 + \left(\frac{\partial x_2}{\Delta u_2}^{\sigma_1} \right)^2 + \left(\frac{\partial x_3}{\Delta u_2}^{\sigma_1} \right)^2 \right) \\
&- \left(\frac{\partial x_1}{\Delta u_1} \frac{\partial x_1}{\Delta u_2}^{\sigma_1} + \frac{\partial x_2}{\Delta u_1} \frac{\partial x_2}{\Delta u_2}^{\sigma_1} + \frac{\partial x_3}{\Delta u_1} \frac{\partial x_3}{\Delta u_2}^{\sigma_1} \right)^2 \\
=& \left(\frac{\partial x_1}{\Delta u_1} \right)^2 \left(\frac{\partial x_1}{\Delta u_1}^{\sigma_1} \right)^2 + \left(\frac{\partial x_2}{\Delta u_1} \right)^2 \left(\frac{\partial x_2}{\Delta u_2}^{\sigma_1} \right)^2 + \left(\frac{\partial x_3}{\Delta u_1} \right)^2 \left(\frac{\partial x_3}{\Delta u_2}^{\sigma_1} \right)^2 \\
&+ \left(\frac{\partial x_1}{\Delta u_1} \right)^2 \left(\frac{\partial x_2}{\Delta u_2}^{\sigma_1} \right)^2 + \left(\frac{\partial x_1}{\Delta u_1} \right)^2 \left(\frac{\partial x_3}{\Delta u_2}^{\sigma_1} \right)^2 + \left(\frac{\partial x_2}{\Delta u_1} \right)^2 \left(\frac{\partial x_1}{\Delta u_2}^{\sigma_1} \right)^2 \\
&+ \left(\frac{\partial x_2}{\Delta u_1} \right)^2 \left(\frac{\partial x_3}{\Delta u_2}^{\sigma_1} \right)^2 + \left(\frac{\partial x_3}{\Delta u_1} \right)^2 \left(\frac{\partial x_1}{\Delta u_2}^{\sigma_1} \right)^2 + \left(\frac{\partial x_3}{\Delta u_1} \right)^2 \left(\frac{\partial x_2}{\Delta u_2}^{\sigma_1} \right)^2 \\
&- \left(\frac{\partial x_1}{\Delta u_1} \right)^2 \left(\frac{\partial x_1}{\Delta u_2}^{\sigma_1} \right)^2 - \left(\frac{\partial x_2}{\Delta u_1} \right)^2 \left(\frac{\partial x_2}{\Delta u_2}^{\sigma_1} \right)^2 - \left(\frac{\partial x_3}{\Delta u_1} \right)^2 \left(\frac{\partial x_3}{\Delta u_2}^{\sigma_1} \right)^2 \\
&-2 \frac{\partial x_1}{\Delta u_1} \frac{\partial x_2}{\Delta u_1} \frac{\partial x_1}{\Delta u_2}^{\sigma_1} \frac{\partial x_2}{\Delta u_2}^{\sigma_1} - 2 \frac{\partial x_1}{\Delta u_1} \frac{\partial x_1}{\Delta u_2}^{\sigma_1} \frac{\partial x_3}{\Delta u_1} \frac{\partial x_3}{\Delta u_2}^{\sigma_1} \\
&-2 \frac{\partial x_2}{\Delta u_1} \frac{\partial x_2}{\Delta u_2}^{\sigma_1} \frac{\partial x_3}{\Delta u_1} \frac{\partial x_3}{\Delta u_2}^{\sigma_1} \\
=& \left(\frac{\partial x_2}{\Delta u_1} \frac{\partial x_3}{\Delta u_2}^{\sigma_1} - \frac{\partial x_3}{\Delta u_1} \frac{\partial x_2}{\Delta u_2}^{\sigma_1} \right)^2 + \left(\frac{\partial x_1}{\Delta u_2}^{\sigma_1} \frac{\partial x_3}{\Delta u_1} - \frac{\partial x_1}{\Delta u_1} \frac{\partial x_3}{\Delta u_2}^{\sigma_1} \right)^2 + \left(\frac{\partial x_1}{\Delta u_1} \frac{\partial x_2}{\Delta u_2}^{\sigma_1} - \frac{\partial x_1}{\Delta u_2}^{\sigma_1} \frac{\partial x_2}{\Delta u_1} \right)^2 \\
=& \left| \frac{\partial x}{\Delta u_1} \times \frac{\partial x}{\Delta u_2}^{\sigma_1} \right|^2 .
\end{aligned}$$

This completes the proof.

Theorem 2.1 combined with the elementary properties of the rank of a matrix, gives the following equivalence.

Corollary 2.1. *Let $x : U \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$ be σ_1-completely delta differentiable. At each point of U the following conditions hold.*

1. *The vectors* $\frac{\partial x}{\Delta u_1}$ *and* ${\frac{\partial x}{\Delta u_2}}^{\sigma_1}$ *are independent.*
2. *The* σ_1*-Jacobian matrix* M *has rank* 2.
3. *There exist* $i, j \in \{1,2,3\}$ *so that*

$$\det \begin{pmatrix} \frac{\partial x_i}{\Delta u_1} & {\frac{\partial x_i}{\Delta u_2}}^{\sigma_1} \\ \frac{\partial x_j}{\Delta u_1} & {\frac{\partial x_j}{\Delta u_2}}^{\sigma_1} \end{pmatrix} \neq 0.$$

4. $\frac{\partial x}{\Delta u_1} \times {\frac{\partial x}{\Delta u_2}}^{\sigma_1} \neq 0.$
5. $\det G > 0.$

Corollary 2.1 motivates us to give the following definition.

Definition 2.2. A surface S is said to be σ_1-regular at a point of U if the conditions of Corollary 2.1 hold at that point. The surface S is said to be σ_1-regular on U if it is σ_1-regular at each point of U.

Now, we will give definitions for a smooth map, a σ_1-homomorphism and σ_1-diffeomorphism.

Definition 2.3. We say that a map $g : W_1 \to W_2$ is a smooth map if it is continuous and its Δ_1 and Δ_2 partial derivatives exist and are continuous on W_1.

Definition 2.4. We say that a map $g : W_2 \to W_1$ is a smooth map if it is continuous and its $\Delta_{(1)}$, $\Delta_{(2)}$ and $\Delta_{(3)}$ partial derivatives exist and are continuous on W_2.

Definition 2.5. We say that a map $g : W_1 \to W_2$ is a time scale σ_1-homeomorphism(shortly σ_1-homeomorphism) if it has the following properties.

1. g is a bijection, i.e., one-to-one and onto.
2. g is continuous.
3. the inverse map g^{-1} exists and it is continuous.
4. If

$$g = (g_1, g_2, g_3), \quad g^{-1} = (G_1, G_2),$$

then

$$g_j(\sigma_1(u_1), u_2) = g_j(u_1, \sigma_2(u_2)) = \sigma_{(j)}(g_j(u_1, u_2)), \quad j \in \{1,2,3\}, \tag{2.1}$$

$(u_1, u_2) \in W_1$, and

$$\begin{aligned} G_j(\sigma_{(1)}(u_{(1)}), u_{(2)}, u_{(3)}) &= G_j(u_{(1)}, \sigma_{(2)}(u_{(2)}), u_{(3)}) \\ &= G_j(u_{(1)}, u_{(2)}, \sigma_{(3)}(u_{(3)})) \\ &= \sigma_j(G_j(u_{(1)}, u_{(2)}, u_{(3)})), \quad j \in \{1,2\}, \end{aligned} \tag{2.2}$$

$(u_{(1)}, u_{(2)}, u_{(3)}) \in W_2$.

Definition 2.6. A map $g : W_1 \to W_2$ that satisfies (2.1) will be called $\sigma_1\sigma_2\sigma_{(1)}\sigma_{(2)}\sigma_{(3)}$-map.

Definition 2.7. A map $g : W_2 \to W_1$ that satisfies (2.2) will be called $\sigma_{(1)}\sigma_{(2)}\sigma_{(3)}\sigma_1\sigma_2$-map.

Definition 2.8. A map $g : W_1 \to W_2$ is called a time scale σ_1-diffeomorphism(shortly σ_1-diffeomorphism) if the has the following properties.

1. g is a σ_1-homomorphism.
2. g and g^{-1} are smooth maps.

Suppose that S is a smooth surface given by $x = x(u)$, $u \in U$ and

$$u = (u_{(1)}, u_{(2)}, u_{(3)}) : \widetilde{U} \to U$$

be a σ_1-diffeomorphism of $\widetilde{U}$ onto U. Set

$$\widetilde{u} = (u_{(1)}, u_{(2)}, u_{(3)}) \in \widetilde{U}.$$

Definition 2.9. We say that the surface $\widetilde{S}$ defined by

$$x = x(u(\widetilde{u})), \quad \widetilde{u} \in \widetilde{U},$$

is obtained from S by a change of parameters.

Definition 2.10. We say that a property of the surface S is independent of parameters if it holds at corresponding points of all surfaces $\widetilde{S}$ obtained from S by a change of parameters.

We have

$$\begin{aligned} x &= x(u(\widetilde{u})) \\ &= (x_1(u(\widetilde{u})), x_2(u(\widetilde{u})), x_3(u(\widetilde{u}))) \\ &= (x_1(u_1(u_{(1)}, u_{(2)}, u_{(3)}), u_2(u_{(1)}, u_{(2)}, u_{(3)})), \\ &\quad x_2(u_1(u_{(1)}, u_{(2)}, u_{(3)}), u_2(u_{(1)}, u_{(2)}, u_{(3)})), \end{aligned}$$

$$x_3(u_1(u_{(1)},u_{(2)},u_{(3)}),u_2(u_{(1)},u_{(2)},u_{(3)}))), \quad (u_{(1)},u_{(2)},u_{(3)}) \in \widetilde{U}.$$

For the partial derivatives of x_j with respect to the variables $u_{(l)}$, $j,l \in \{1,2\}$, we have the following representation

$$\frac{\partial x_j}{\Delta u_{(l)}} = \frac{\partial x_j}{\Delta u_1}\frac{\partial u_1}{\Delta u_{(l)}} + \frac{\partial x_j}{\Delta u_2}^{\sigma_1}\frac{\partial u_2}{\Delta u_{(l)}}, \quad j,l \in \{1,2\}.$$

Let

$$P = \begin{pmatrix} \frac{\partial u_1}{\Delta u_{(1)}} & \frac{\partial u_1}{\Delta u_{(2)}} \\ \frac{\partial u_2}{\Delta u_{(1)}} & \frac{\partial u_2}{\Delta u_{(2)}} \end{pmatrix}.$$

Then

$$\widetilde{M} = MP,$$

whereupon

$$\begin{aligned} \widetilde{G} &= \widetilde{M}^T\widetilde{M} \\ &= P^T M^T MP \\ &= P^T GP. \end{aligned} \tag{2.3}$$

Hence,

$$\begin{aligned} \det\widetilde{G} &= \det(P^T GP) \\ &= (\det P)^2 \det G. \end{aligned}$$

By the last equation, it follows that the property S being σ_1-regular at a point is independent of parameters.

Let now, U_1 be a subdomain of U so that $\overline{U_1} \subset U$, where $\overline{U_1}$ is the closure of U_1. With Σ we will denote the restriction of the surface S given by $x = x(u)$, $u \in \Sigma$.

Definition 2.11. Define the area of Σ to be

$$A(\Sigma) = \int\int_{U_1} \sqrt{\det G}\Delta u_1 \Delta u_2.$$

If $u = u(\widetilde{u})$ is a change of parameters and $\widetilde{U}_1$ maps U_1, then

$$A(\widetilde{\Sigma}) = \int\int_{\widetilde{U}_1} \sqrt{\det\widetilde{G}}\Delta\widetilde{u}_1\Delta\widetilde{u}_2$$

$$= \int\int_{\widetilde{U}_1} \sqrt{\det G}|\det P|\Delta\widetilde{u}_1\Delta\widetilde{u}_2$$

$$= \int\int_{U} \sqrt{\det G}\Delta u_1\Delta u_2$$

$$= A(\Sigma).$$

Thus, the area of a surface is independent of variables.

Let j and l be any two fixed integers from 1 to 3. Let also, U be a domain in the x_j, x_l-plane. The equation

$$x_k = f_k(x_j, x_l), \quad k \in \{1,2,3\}, \quad k \neq j, \quad k \neq l,'$$

define a surface in $\mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$.

Definition 2.12. A surface defined in this way will be said to be given nonparametric or explicit form.

In order for a surface to be expressible in nonparametric form, it is necessary for the projection map

$$(x_1, x_2, x_3) \to (x_j, x_l)$$

to be one-to-one. This is not true in the general case for the whole surface, but we have the following important theorem.

Theorem 2.2. *Let S be a surface given by $x = x(u)$ and let $a \in \mathbb{T}_1 \times \mathbb{T}_2$ be a point at which S is regular. Then there is a neighbourhood Δ of a such that the surface Σ obtained by the restriction $x(u)$ to Δ has a parametrization $\widetilde{\Sigma}$ in nonparametric form.*

Proof. By Corollary 2.1 3) and the inverse image theorem, we conclude that there exists a neighbourhood Δ of a in which the map $(u_1, u_2) \to (x_j, x_l)$ is a σ_1-diffeomorphism. If $x = x(u)$ is a smooth map, then the inverse map $(x_1, x_2) \to (u_1, u_2)$ is also smooth and the same is true of the composed map

$$(x_j, x_l) \to (u_1, u_2) \to (x_1, x_2, x_3), \tag{2.4}$$

which defines $\widetilde{\Sigma}$. This completes the proof.

By Theorem 2.2, we conclude that when we study the local behaviour of a surface, for convenience, we can assume that the surface is in nonparametric form. Note that the parametrization (2.4) shows that in a neighbourhood of a σ_1-regular point the mapping $x = x(u)$ is always one-to-one.

In order to study more closely the behaviour of a surface near a given point, we consider the set of curves passing through the point and lying on the surface.

Definition 2.13. By a curve C in $\mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$ we mean a continuously differentiable map

$$\phi : [\alpha, \beta] \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}, \tag{2.5}$$

where $[\alpha, \beta] \subset \mathbb{T}$. We will use the notation

$$\begin{aligned} \phi &= \phi(t) \\ &= (\phi_1(t), \phi_2(t), \phi_3(t)), \quad t \in [\alpha, \beta]. \end{aligned}$$

Definition 2.14. The tangent vector of the curve C at the point $t_0 \in [\alpha, \beta]$ is the vector

$$\phi^{\Delta}(t_0) = (\phi_1^{\Delta}(t_0), \phi_2^{\Delta}(t_0), \phi_3^{\Delta}(t_0)).$$

The curve C is said to be regular at t_0 if $\phi^{\Delta}(t_0) \neq 0$. The curve C is said to be regular in $[\alpha, \beta]$ if it is regular at each point of $[\alpha, \beta]$.

Now, suppose that S is a surface defined by $x = x(u)$, $u \in U$, and a curve C is given by (2.5).

Definition 2.15. We say that C lies on S if

$$\phi([\alpha, \beta]) \subset x(U).$$

Since we are interested in local study of S, we choose any point a at which S is regular, and let us restrict $x(a)$ to a neighbourhood given in Theorem 2.2. We shall denote this restricted domain by U, and the surface by S. Then, we have the representation (2.4) and the mapping $x = x(u)$ is one-to-one in U. Consider the set of all curves C which lie on S and pass through the point $b = x(a)$. Assume that there is a point $t_0 \in [\alpha, \beta]$ such that for each curve C we have

$$\phi(t_0) = b.$$

By (2.4), it follows that each such curve corresponds to a curve $u = u(t)$ in U for which $u(t_0) = a$. Conversely, to each curve $u = u(t)$ in U with $u(t_0) = a$ corresponds a curve $\phi(t) = x(u(t))$ on S with $\phi(t_0) = b$. Then, for the tangent of C we have

$$\phi^{\Delta}(t_0) = \frac{\partial x}{\Delta u_1}(a) u_1^{\Delta}(t_0) + \frac{\partial x}{\Delta u_2}^{\sigma_1}(a) u_2^{\Delta}(t_0).$$

Theorem 2.3. *At a regular point of a surface S, if we consider the set of all curves which lie on S and pass through this point, then the set of their tangent vectors at the point forms a two dimensional vector space.*

Proof. Since we can find curves $u = u(t)$ in U such that

$$u(t_0) = a$$

and $u_1^{\Delta}(t_0)$, $u_2^{\Delta}(t_0)$ take arbitrarily assigned values, it follows that the set of tangent vectors $x^{\Delta}(t_0)$ consists of all linear combinations of the two vectors

$$\frac{\partial x}{\Delta u_1} \quad \text{and} \quad \frac{\partial x}{\Delta u_2}^{\sigma_1}.$$

By Corollary 2.1, it follows that these vectors are independent and therefore span a two dimensional vector space. This completes the proof.

Definition 2.16. The vector space described in Theorem 2.3 is called the tangent plane to the surface S at the point $b = x(a)$ and it is denoted by Π or $\Pi(a)$.

Therefore, a surface S has at every regular point a tangent plane, which is independent of parameters. For the length of ϕ^Δ, we have

$$\begin{aligned}
|\phi^\Delta|^2 &= \phi^\Delta \cdot \phi^\Delta \\
&= \left(\frac{\partial x}{\Delta u_1} u_1^\Delta + \frac{\partial x}{\Delta u_2}^{\sigma_1} u_2^\Delta\right) \cdot \left(\frac{\partial x}{\Delta u_1} u_1^\Delta + \frac{\partial x}{\Delta u_2}^{\sigma_1} u_2^\Delta\right) \\
&= \left(\frac{\partial x}{\Delta u_1}\right)^2 u_1^\Delta u_1^\Delta + 2\left(\frac{\partial x}{\Delta u_1}\right)\left(\frac{\partial x}{\Delta u_2}^{\sigma_1}\right) u_1^\Delta u_2^\Delta + \left(\frac{\partial x}{\Delta u_2}^{\sigma_1}\right)^2 u_2^\Delta u_2^\Delta \\
&= \sum_{i,j=1}^{2} F_{1ij} u_i^\Delta u_j^\Delta,
\end{aligned}$$

where

$$\begin{aligned}
F_{111} &= \left(\frac{\partial x}{\Delta u_1}\right)^2, \\
F_{112} &= F_{121} \\
&= \frac{\partial x}{\Delta u_1} \frac{\partial x}{\Delta u_2}^{\sigma_1}, \\
F_{122} &= \left(\frac{\partial x}{\Delta u_2}^{\sigma_1}\right)^2,
\end{aligned}$$

i.e., the length of ϕ^Δ is expressed with a quadratic form with matrix (F_{1ij}). This quadratic form is referred to as the first fundamental form of the surface S. The determinant of this form defines areas on this surface. The length of the curves on the surface S are obtained by

$$L = \int_\alpha^\beta |\phi^\Delta(t)| \Delta t.$$

For an arbitrary curve C, denote

$$s(t_0) = \int_{\alpha}^{t_0} |\phi^{\Delta}(t)|\Delta t.$$

We have

$$s^{\Delta}(t_0) = \phi^{\Delta}(t_0) \geq 0, \quad t_0 \in [\alpha, \beta],$$

and we have the monotone mapping

$$s : [\alpha, \beta] \to [0, L]. \tag{2.6}$$

If the curve C is regular, then

$$s^{\Delta}(t_0) = |\phi^{\Delta}(t_0)| > 0$$

and the map (2.6) has a differentiable inverse $t = t(s)$. The composed map

$$\widetilde{\phi}(s) : [0, L] \overset{t(s)}{\to} [\alpha, \beta] \overset{\phi(t)}{\to} \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$$

defines a curve $\widetilde{C}$ which is called the parametrization of C with respect to the arc length.

Now, suppose that S is a surface given by $x = x(u)$, $x \in \mathscr{C}^2(U)$, and let S be σ_1-regular at the point $b = x(a)$. Let Π be the tangent plane to S at b and let $\Pi^{\perp}$ be its orthogonal complement which is a 1-dimensional space called normal space to S at the point. Each vector is determined by its projections on Π and $\Pi^{\perp}$.

Definition 2.17. An arbitrary vector $N \in \Pi^{\perp}$ is called a normal to S.

A normal vector N to S is orthogonal to $\frac{\partial x}{\Delta u_1}$ and $\frac{\partial x}{\Delta u_2}^{\sigma_1}$. Suppose that

$$x = x(u_1(s), u_2(s)),$$

$$u_j(\sigma(s)) = \sigma_j(u_j(s)), \quad s \in \mathbb{T}, \quad j \in \{1, 2\},$$

and σ_1 is Δ_1-differentiable on $\mathbb{T}_1$. Then

$$\frac{dx}{\Delta s} = \frac{\partial x}{\Delta u_1}\frac{du_1}{\Delta s} + \frac{\partial x}{\Delta u_2}^{\sigma_1}\frac{du_2}{\Delta s},$$

$$\begin{aligned}
\frac{d^2x}{\Delta s^2} &= \frac{d}{\Delta s}\left(\frac{\partial x}{\Delta u_1}\frac{du_1}{\Delta s} + \frac{\partial x}{\Delta u_2}^{\sigma_1}\frac{du_2}{\Delta s}\right) \\
&= \frac{d^2u_1}{\Delta s^2}\frac{\partial x}{\Delta u_1} + \frac{du_1}{\Delta s}^{\sigma}\left(\frac{\partial^2 x}{\Delta u_1^2}\frac{du_1}{\Delta s} + \frac{\partial^2 x}{\Delta u_1 \Delta u_2}^{\sigma_1}\frac{du_2}{\Delta s}\right) \\
&\quad + \frac{d^2u_2}{\Delta s^2}\frac{\partial x}{\Delta u_2}^{\sigma_1} + \frac{du_2}{\Delta s}^{\sigma}\left(\frac{\partial}{\Delta u_1}\left(\frac{\partial x}{\Delta u_2}^{\sigma_1}\right)\frac{du_1}{\Delta s} + \frac{\partial}{\Delta u_2}\left(\frac{\partial x}{\Delta u_2}^{\sigma_1}\right)^{\sigma_1}\frac{du_2}{\Delta s}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{d^2u_1}{\Delta s^2}\frac{\partial x}{\Delta u_1} + \frac{d^2u_2}{\Delta s^2}\frac{\partial x}{\Delta u_2}^{\sigma_1} \\
&\quad + \frac{du_1}{\Delta s}^{\sigma}\frac{du_1}{\Delta s}\frac{\partial^2 x}{\Delta u_1^2} + \frac{du_1}{\Delta s}^{\sigma}\frac{du_2}{\Delta s}\frac{\partial^2 x}{\Delta u_1 \Delta u_2}^{\sigma_1} \\
&\quad + \frac{du_1}{\Delta s}\frac{du_2}{\Delta s}^{\sigma}\frac{\partial}{\Delta u_1}\left(\frac{\partial x}{\Delta u_2}^{\sigma_1}\right) + \frac{\partial^2 x}{\Delta u_2^2}^{\sigma_1\sigma_1}\frac{du_2}{\Delta s}^{\sigma}\frac{du_2}{\Delta s}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{d^2x}{\Delta s^2}N &= \frac{d^2u_1}{\Delta s^2}\left(\frac{\partial x}{\Delta u_1}N\right) + \frac{d^2u_2}{\Delta s^2}\left(\frac{\partial x}{\Delta u_2}^{\sigma_1}N\right) \\
&\quad + \frac{du_1}{\Delta s}^{\sigma}\frac{du_1}{\Delta s}\left(\frac{\partial^2 x}{\Delta u_1^2}N\right) + \frac{du_1}{\Delta s}^{\sigma}\frac{du_2}{\Delta s}\left(\frac{\partial^2 x}{\Delta u_1 \Delta u_2}^{\sigma_1}N\right) \\
&\quad + \frac{du_1}{\Delta s}\frac{du_2}{\Delta s}^{\sigma}\left(\frac{\partial}{\Delta u_1}\left(\frac{\partial x}{\Delta u_2}^{\sigma_1}\right)N\right) + \left(\frac{\partial^2 x}{\Delta u_2^2}^{\sigma_1\sigma_1}N\right)\frac{du_2}{\Delta s}^{\sigma}\frac{du_2}{\Delta s} \\
&= \frac{du_1}{\Delta s}^{\sigma}\frac{du_1}{\Delta s}\left(\frac{\partial^2 x}{\Delta u_1^2}N\right) + \frac{du_1}{\Delta s}^{\sigma}\frac{du_2}{\Delta s}\left(\frac{\partial^2 x}{\Delta u_1 \Delta u_2}^{\sigma_1}N\right) \\
&\quad + \frac{du_1}{\Delta s}\frac{du_2}{\Delta s}^{\sigma}\left(\frac{\partial}{\Delta u_1}\left(\frac{\partial x}{\Delta u_2}^{\sigma_1}\right)N\right) + \left(\frac{\partial^2 x}{\Delta u_2^2}^{\sigma_1\sigma_1}N\right)\frac{du_2}{\Delta s}^{\sigma}\frac{du_2}{\Delta s} \\
&= b_{11}(N)\frac{du_1}{\Delta s}^{\sigma}\frac{du_1}{\Delta s} + b_{12}(N)\frac{du_1}{\Delta s}^{\sigma}\frac{du_2}{\Delta s} \\
&\quad + b_{21}(N)\frac{du_1}{\Delta s}\frac{du_2}{\Delta s}^{\sigma} + b_{22}(N)\frac{du_2}{\Delta s}^{\sigma}\frac{du_2}{\Delta s},
\end{aligned}
$$

where

$$
\begin{aligned}
b_{11}(N) &= \frac{\partial^2 x}{\Delta u_1^2}N, \\
b_{12}(N) &= \frac{\partial^2 x}{\Delta u_2 \Delta u_1}^{\sigma_1}N, \\
b_{21}(N) &= \frac{\partial}{\Delta u_1}\left(\frac{\partial x}{\Delta u_2}^{\sigma_1}\right)N, \\
b_{22}(N) &= \frac{\partial^2 x}{\Delta u_2^2}^{\sigma_1\sigma_1}N.
\end{aligned}
$$

Note that

$$\left(\frac{ds}{\Delta t}\right)^2 = |x^\Delta(t)|^2$$

$$= \sum_{i,j=1}^{2} F_{1ij} u_i^\Delta u_j^\Delta,$$

$$\frac{du_i}{\Delta s} = u_i^\Delta \frac{1}{\frac{ds}{\Delta t}}, \quad i \in \{1,2\}.$$

Hence,

$$\frac{d^2x}{\Delta s^2} N = b_{11}(N) \frac{du_1}{\Delta s}^\sigma \frac{du_1}{\Delta s} + b_{12}(N) \frac{du_1}{\Delta s}^\sigma \frac{du_2}{\Delta s}$$

$$+ b_{21}(N) \frac{du_1}{\Delta s} \frac{du_2}{\Delta s}^\sigma + b_{22}(N) \frac{du_2}{\Delta s}^\sigma \frac{du_2}{\Delta s}$$

$$= b_{11}(N) \frac{u_1^{\Delta\sigma} u_1^\Delta}{\frac{ds}{\Delta t}^\sigma \frac{ds}{\Delta t}} + b_{12}(N) \frac{u_1^{\Delta\sigma} u_2^\Delta}{\frac{ds}{\Delta t}^\sigma \frac{ds}{\Delta t}}$$

$$+ b_{21}(N) \frac{u_1^\Delta u_2^{\Delta\sigma}}{\frac{ds}{\Delta t}^\sigma \frac{ds}{\Delta t}} + b_{22}(N) \frac{u_2^\Delta u_2^{\Delta\sigma}}{\frac{ds}{\Delta t}^\sigma \frac{ds}{\Delta t}}$$

$$= \frac{1}{\frac{ds}{\Delta t}^\sigma \frac{ds}{\Delta t}} \Big(b_{11}(N) u_1^{\Delta\sigma} u_1^\Delta + b_{12}(N) u_1^{\Delta\sigma} u_2^\Delta$$

$$+ b_{21}(N) u_1^\Delta u_2^{\Delta\sigma} + b_{22}(N) u_2^\Delta u_2^{\Delta\sigma} \Big).$$

Definition 2.18. The quadratic form

$$b_{11}(N) u_1^{\Delta\sigma} u_1^\Delta + b_{12}(N) u_1^{\Delta\sigma} u_2^\Delta + b_{21}(N) u_1^\Delta u_2^{\Delta\sigma} + b_{22}(N) u_2^\Delta u_2^{\Delta\sigma}$$

is said to be the second fundamental form of S with respect to N.

Set

$$T = \frac{dx}{\Delta s},$$

$$k(N,T) = \frac{b_{11}(N) u_1^{\Delta\sigma} u_1^\Delta + b_{12}(N) u_1^{\Delta\sigma} u_2^\Delta + b_{21}(N) u_1^\Delta u_2^{\Delta\sigma} + b_{22}(N) u_2^\Delta u_2^{\Delta\sigma}}{\sum_{i,j=1}^{2} F_{1ij} u_i^\Delta u_j^\Delta}.$$

Definition 2.19. $k(N,T)$ is said to be the normal curvature of S in the direction T with respect to the normal vector N.

Define

$$k_1(N) = \max_T k(N,T),$$

$$k_2(N) = \min_T k(N,T).$$

Definition 2.20. k_1 and k_2 are said to be the principal curvatures of S at the point with respect to the normal N. The average mean

$$H(N) = \frac{k_1(N) + k_2(N)}{2}$$

is called the mean curvature of S at the point with respect to the normal N.

Note that $k_1(N)$ and $k_2(N)$ are the roots of the equation

$$\det(b_{ij}(N) - \lambda F_{1ij}) = 0,$$

or

$$\det(F_{1ij})\lambda^2 - (F_{122}b_{11}(N) + F_{111}b_{22}(N) - F_{112}b_{21}(N) - F_{121}b_{12}(N))\lambda + \det(b_{ij}(N)) = 0.$$

Thus,

$$H(N) = \frac{F_{122}b_{11}(N) + F_{111}b_{22}(N) - F_{112}b_{21}(N) - F_{121}b_{12}(N)}{2\det(F_{1ij})}.$$

Observe that $b_{ij}(N)$, $i,j \in \{1,2\}$, are linear in N and then $H(N)$ is linear in N. Therefore

$$H(N) = HN.$$

Definition 2.21. The vector H is said to be the mean curvature vector of S at the point.

If E is a unit vector of Π, then

$$H = H(E)E.$$

Definition 2.22. A surface S is said to be a minimal surface if its mean curvature vector H vanishes at every point.

Suppose that S is a minimal surface. Then $H = 0$ if and only if $H(N) = 0$ for any $N \in \Pi^{\perp}$. Thus, a minimal surface is characterized in terms of their first and second fundamental forms by the equation

$$F_{122}b_{11}(N) + F_{111}b_{22}(N) - F_{112}b_{21}(N) - F_{121}b_{12}(N) = 0. \tag{2.7}$$

2.2 Nonparametric Surfaces

Suppose that σ_1 is Δ_1-differentiable on $\mathbb{T}_1$. In this section, we consider surfaces in nonparametric forms. Assume that the surface S is defined by

$$x_3 = f(x_1, x_2), \quad (x_1, x_2) \in \mathbb{T}_1 \times \mathbb{T}_2,$$

or equivalently

$$\begin{aligned} x_1 &= u_1 \\ x_2 &= u_2 \\ x_3 &= f(u_1, u_2), \quad (u_1, u_2) \in \mathbb{T}_1 \times \mathbb{T}_2. \end{aligned} \tag{2.8}$$

Thus,

$$\begin{aligned} x &= x(u) \\ &= x(u_1, u_2) \\ &= (x_1(u_1, u_2), x_2(u_1, u_2), f(u_1, u_2)), \quad (u_1, u_2) \in \mathbb{T}_1 \times \mathbb{T}_2. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial x}{\Delta u_1} &= \left(1, 0, \frac{\partial f}{\Delta u_1}\right), \\ \frac{\partial x}{\Delta u_2} &= \left(0, 1, \frac{\partial f}{\Delta u_2}\right), \\ \frac{\partial x}{\Delta u_2}^{\sigma_1} &= \left(0, 1, \frac{\partial f}{\Delta u_2}^{\sigma_1}\right), \\ F_{111} &= \left(\frac{\partial x}{\Delta u_1}\right)^2 \\ &= 1 + \left(\frac{\partial f}{\Delta u_1}\right)^2, \\ F_{112} &= \frac{\partial x}{\Delta u_1} \frac{\partial x}{\Delta u_2}^{\sigma_1} \\ &= \frac{\partial f}{\Delta u_1} \frac{\partial f}{\Delta u_2}^{\sigma_1}, \end{aligned}$$

$$F_{122} = \left(\frac{\partial x}{\Delta u_2}^{\sigma_1}\right)^2$$
$$= 1 + \left(\frac{\partial f}{\Delta u_2}^{\sigma_1}\right)^2.$$

Remark 2.1. Note that the vectors $\frac{\partial x}{\Delta u_1}$ and $\frac{\partial x}{\Delta u_2}^{\sigma_1}$ are obviously linearly independent.

Theorem 2.4. *Let $N_3 \in \mathbb{R}$ be arbitrarily chosen and fixed. Then there exist unique N_1, N_2 such that*

$$N = (N_1, N_2, N_3) \in \Pi^{\perp}.$$

Proof. Observe that the vector $N \in \Pi^{\perp}$ if and only if

$$\begin{aligned} N \cdot \frac{\partial x}{\Delta u_1} &= 0 \\ N \cdot \frac{\partial x}{\Delta u_2}^{\sigma_1} &= 0. \end{aligned} \tag{2.9}$$

Note that

$$N \cdot \frac{\partial x}{\Delta u_1} = (N_1, N_2, N_3) \cdot \left(1, 0, \frac{\partial f}{\Delta u_1}\right)$$
$$= N_1 + N_3 \frac{\partial f}{\Delta u_1},$$
$$N \cdot \frac{\partial x}{\Delta u_2}^{\sigma_1} = (N_1, N_2, N_3)\left(0, 1, \frac{\partial f}{\Delta u_2}^{\sigma_1}\right)$$
$$= N_2 + N_3 \frac{\partial f}{\Delta u_3}.$$

Hence and the system (2.9), we find

$$N_1 = -N_3 \frac{\partial f}{\Delta u_1}$$
$$N_2 = -N_3 \frac{\partial f}{\Delta u_2}^{\sigma_1},$$

i.e.,

$$N = N_3 \left(-\frac{\partial f}{\Delta u_1}, -\frac{\partial f}{\Delta u_2}^{\sigma_1}, 1\right).$$

This completes the proof.

Corollary 2.2. *Let S be defined by $x = x(u)$, $u \in U$, $x \in \mathscr{C}^2(U)$, $b = x(a)$, $a \in U$, be a σ_1-regular point of S and let N be a normal to S at this point. Then there exists a neighbourhood Q of a and $N(u) \in \mathscr{C}^1(Q)$ such that $N(u) \in \Pi^{\perp}(u)$ and $N(a) = N$.*

Proof. By Theorem 2.2, it follows that there is a neighbourhood Q of a in which the surface S has a parametrization in the form (2.8). Let

$$N = (N_1, N_2, N_3),$$

where

$$N_1 = -N_3 \frac{\partial f}{\Delta u_1},$$

$$N_2 = -N_3 \frac{\partial f}{\Delta u_2}^{\sigma_1}, \quad N_3 \in \mathbb{R}.$$

Then, $N(u)$ has the desired properties. This completes the proof.

Next, we deduce

$$\frac{\partial^2 x}{\Delta u_i \Delta u_j} = \left(0, 0, \frac{\partial^2 f}{\Delta u_i \Delta u_j}\right), \quad i, j \in \{1, 2\}.$$

For any normal $N = (N_1, N_2, N_3)$, we have

$$b_{11}(N) = \frac{\partial^2 f}{\Delta u_1^2} N_3,$$

$$b_{12}(N) = \frac{\partial}{\Delta u_2} \left(\frac{\partial f}{\Delta u_1}\right)^{\sigma_1} N_3,$$

$$b_{21}(N) = \frac{\partial}{\Delta u_1} \left(\frac{\partial f}{\Delta u_2}^{\sigma_1}\right) N_3,$$

$$b_{22}(N) = \frac{\partial^2 f}{\Delta u_2^2}^{\sigma_1 \sigma_1} N_3.$$

Then, the equation (2.7) for a minimal surface takes the form

$$\left(1 + \left(\frac{\partial f}{\Delta u_2}^{\sigma_1}\right)^2\right) \frac{\partial^2 f}{\Delta u_1^2} N_3 + \left(1 + \left(\frac{\partial f}{\Delta u_1}\right)^2\right) \frac{\partial^2 f}{\Delta u_2^2}^{\sigma_1 \sigma_1} N_3$$
$$- \frac{\partial f}{\Delta u_1} \frac{\partial f}{\Delta u_2}^{\sigma_1} \frac{\partial}{\Delta u_1} \left(\frac{\partial f}{\Delta u_2}\right)^{\sigma_1} N_3 - \frac{\partial f}{\Delta u_1} \frac{\partial f}{\Delta u_2}^{\sigma_1} \frac{\partial^2 f}{\Delta u_2 \Delta u_1}^{\sigma_1} N_3 = 0,$$

or

$$\left(1+\left(\frac{\partial f}{\Delta u_2}^{\sigma_1}\right)^2\right)\frac{\partial^2 f}{\Delta u_1^2}+\left(1+\left(\frac{\partial f}{\Delta u_1}\right)^2\right)\frac{\partial^2 f}{\Delta u_2^2}^{\sigma_1\sigma_1}$$
$$-\frac{\partial f}{\Delta u_1}\frac{\partial f}{\Delta u_2}^{\sigma_1}\frac{\partial}{\Delta u_1}\left(\frac{\partial f}{\Delta u_2}\right)^{\sigma_1}-\frac{\partial f}{\Delta u_1}\frac{\partial f}{\Delta u_2}^{\sigma_1}\frac{\partial^2 f}{\Delta u_2\Delta u_1}^{\sigma_1}=0. \tag{2.10}$$

Example 2.1. Let $\mathbb{T}_1=\mathbb{T}_2=\mathbb{Z}$, $\mathbb{T}_{(1)}=\mathbb{T}_{(2)}=\mathbb{T}_{(3)}=\mathbb{R}$ and

$$x(u)=(u_1,u_2,-u_1^2-3u_1u_2+u_2^2),\quad (u_1,u_2)\in\mathbb{T}_1\times\mathbb{T}_2.$$

Here

$$\sigma_1(u_1)=u_1+1,$$

$$\sigma_2(u_2)=u_2+1,$$

$$f(u_1,u_2)=-u_1^2-3u_1u_2+u_2^2,\quad (u_1,u_2)\in\mathbb{T}_1\times\mathbb{T}_1.$$

Then

$$\begin{aligned}\frac{\partial f}{\Delta u_1}(u_1,u_2)&=-\sigma_1(u_1)-u_1-3u_2\\&=-u_1-1-u_1-3u_2\\&=-2u_1-3u_2-1,\\ \frac{\partial f}{\Delta u_2}(u_1,u_2)&=\sigma_2(u_2)+u_2-3u_1\\&=u_2+1+u_2-3u_1\\&=2u_2-3u_1+1,\\ \frac{\partial f}{\Delta u_2}^{\sigma_1}(u_1,u_2)&=2u_2-3\sigma_1(u_1)-1\\&=2u_2-3u_1-3+1\\&=2u_2-3u_1-2,\\ \frac{\partial^2 f}{\Delta u_1\Delta u_2}(u_1,u_2)&=\frac{\partial^2 f}{\Delta u_2\Delta u_1}(u_1,u_2)\end{aligned}$$

$$= -3,$$

$$\frac{\partial^2 f}{\Delta u_1^2}(u_1,u_2) = -2,$$

$$\frac{\partial^2 f}{\Delta u_2^2}(u_1,u_2) = 2,$$

$$\frac{\partial^2 f}{\Delta u_2^2}^{\sigma_1\sigma_1}(u_1,u_2) = 2,$$

$$\frac{\partial}{\Delta u_1}\left(\frac{\partial f}{\Delta u_2}^{\sigma_1}\right)(u_1,u_2) = -3 \quad (u_1,u_2)\in\mathbb{T}_1\times\mathbb{T}_2.$$

Then, the equation for a minimal surface takes the form

$$-2(1+(2u_2-3u_1+1)^2)+2(1+(2u_1-3u_2+1)^2)$$

$$+6(2u_1-3u_2+1)(2u_2-3u_1-2)=0, \quad (u_1,u_2)\in\mathbb{T}_1\times\mathbb{T}_2,$$

or

$$-(2u_2-3u_1-2)^2+(2u_1-3u_2-1)^2+3(2u_1-3u_2-1)(2u_2-3u_1-2)=0, \quad (u_1,u_2)\in\mathbb{T}_1\times\mathbb{T}_2,$$

or

$$-(4u_2^2+9u_1^2+4-12u_1u_2-8u_2+12u_1)$$

$$+(4u_1^2+9u_2^2+1-12u_1u_2-4u_1+6u_2)$$

$$+3(4u_1u_2-6u_1^2-4u_1-6u_2^2+9u_1u_2+6u_2-2u_2+3u_1+2)=0, \quad (u_1,u_2)\in\mathbb{T}_1\times\mathbb{T}_2,$$

or

$$-4u_2^2-9u_1^2-4+12u_1u_2+8u_2-12u_1$$

$$+4u_1^2+9u_2^2+1-12u_1u_2-4u_1+6u_2$$

$$+12u_1u_2-18u_1^2-12u_1-18u_2^2+27u_1u_2+18u_2-6u_2+9u_1+6=0, \quad (u_1,u_2)\in\mathbb{T}_1\times\mathbb{T}_2,$$

or

$$-23u_1^2+39u_1u_2-13u_2^2-19u_1+26u_2+3=0, \quad (u_1,u_2)\in\mathbb{T}_1\times\mathbb{T}_2.$$

2.3 Surfaces that Minimize Area

Let σ_1 be Δ_1-differentiable. Suppose that S is a σ_1-regular surface defined by $x = x(u)$, $u \in U$, $x \in \mathscr{C}^3(U)$. Let Γ be a closed curve which bounds a subdomain Q, and let Σ be the surface defined by $x = x(u)$ restricted to Q. Assume that the area of Σ is less than or equal to the area of any surface $\widetilde{\Sigma}$ defined by $\widetilde{x} = \widetilde{x}(u)$, $u \in Q$, such that $\widetilde{x}(u) = x(u)$ for $u \in \Gamma$. Let $N(u) \in \mathscr{C}^1(U)$ be a normal to S at $x(u)$. Then

$$\begin{aligned} N\frac{\partial x}{\Delta u_1} &= 0 \\ N\frac{\partial x}{\Delta u_2}^{\sigma_1} &= 0. \end{aligned} \tag{2.11}$$

We differentiate the first equation of (2.11) with respect to u_1 and we find

$$N\frac{\partial^2 x}{\Delta u_1^2} + \frac{\partial N}{\Delta u_1}\frac{\partial x}{\Delta u_1}^{\sigma_1} = 0,$$

whereupon

$$\frac{\partial N}{\Delta u_1}\frac{\partial x}{\Delta u_1}^{\sigma_1} = -N\frac{\partial^2 x}{\Delta u_1^2},$$

or

$$\frac{\partial N}{\Delta u_1}\frac{\partial x}{\Delta u_1}^{\sigma_1} = -b_{11}. \tag{2.12}$$

Now, we differentiate the second equation of (2.11) with respect to u_1 and we get

$$\frac{\partial N}{\Delta u_1}\frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1} + N\frac{\partial}{\Delta u_1}\left(\frac{\partial x}{\Delta u_2}^{\sigma_1}\right) = 0,$$

or

$$\begin{aligned} \frac{\partial N}{\Delta u_1}\frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1} &= -N\frac{\partial}{\Delta u_1}\left(\frac{\partial x}{\Delta u_2}^{\sigma_1}\right) \\ &= -b_{21}. \end{aligned} \tag{2.13}$$

Now, we consider an arbitrary function $h \in \mathscr{C}^2(U)$ and for each $\lambda \in \mathbb{R}$, we form the surface

$$S_\lambda : \widetilde{x}(u) = x^{\sigma_1}(u) + \lambda h(u) N(u), \quad u \in U.$$

We find

$$\begin{aligned} \frac{\partial \widetilde{x}}{\Delta u_1} &= \frac{\partial x^{\sigma_1}}{\Delta u_1} + \lambda\left(\frac{\partial h}{\Delta u_1}N + h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right) \\ &= \frac{\partial}{\Delta u_1}\left(x + \mu_1\frac{\partial x}{\Delta u_1}\right) + \lambda\left(\frac{\partial h}{\Delta u_1}N + h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial x}{\Delta u_1} + \frac{\partial \mu_1}{\Delta u_1}\frac{\partial x}{\Delta u_1} + \mu_1^{\sigma_1}\frac{\partial^2 x}{\Delta u_1^2} + \lambda\left(\frac{\partial h}{\Delta u_1}N + h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)\\
&= \left(1+\frac{\partial \mu_1}{\Delta u_1}\right)\frac{\partial x}{\Delta u_1} + \mu_1^{\sigma_1}\frac{\partial^2 x}{\Delta u_1^2} + \lambda\left(\frac{\partial h}{\Delta u_1}N + h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)\\
&= \left(1+\frac{\partial \mu_1}{\Delta u_1}\right)\left(\frac{\partial x}{\Delta u_1}^{\sigma_1} - \mu_1\frac{\partial^2 x}{\Delta u_1^2}\right) + \mu_1^{\sigma_1}\frac{\partial^2 x}{\Delta u_1^2}\\
&\quad +\lambda\left(\frac{\partial h}{\Delta u_1}N + h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)\\
&= \left(1+\frac{\partial \mu_1}{\Delta u_1}\right)\frac{\partial x}{\Delta u_1}^{\sigma_1} + \left(\mu_1^{\sigma_1} - \mu_1 - \mu_1\frac{\partial \mu_1}{\Delta u_1}\right)\frac{\partial^2 x}{\Delta u_1^2}\\
&\quad +\lambda\left(\frac{\partial h}{\Delta u_1}N + h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \widetilde{x}}{\Delta u_2} &= \frac{\partial x}{\Delta u_2}^{\sigma_1} + \lambda\left(\frac{\partial h}{\Delta u_2}N + h^{\sigma_1}\frac{\partial N}{\Delta u_2}\right),\\
\frac{\partial \widetilde{x}}{\Delta u_2}^{\sigma_1} &= \frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1} + \lambda\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1} + h^{\sigma_1\sigma_1}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right).
\end{aligned}
$$

Hence, employing (2.12), we get

$$
\begin{aligned}
\widetilde{F}_{111} &= \left(\frac{\partial \widetilde{x}}{\Delta u_1}^{\sigma_1}\right)^2\\
&= \left(\left(1+\frac{\partial \mu_1}{\Delta u_1}\right)\frac{\partial x}{\Delta u_1}^{\sigma_1} + \left(\mu_1^{\sigma_1} - \mu_1 - \mu_1\frac{\partial \mu_1}{\Delta u_1}\right)\frac{\partial^2 x}{\Delta u_1^2}\right.\\
&\quad \left. +\lambda\left(\frac{\partial h}{\Delta u_1}N + h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)\right)^2\\
&= \left(1+\frac{\partial \mu_1}{\Delta u_1}\right)^2\left(\frac{\partial x}{\Delta u_1}^{\sigma_1}\right)^2 + 2\left(\mu_1^{\sigma_1} - \mu_1 - \mu_1\frac{\partial \mu_1}{\Delta u_1}\right)\left(1+\frac{\partial \mu_1}{\Delta u_1}\right)\frac{\partial x}{\Delta u_1}^{\sigma_1}\frac{\partial^2 x}{\Delta u_1^2}\\
&\quad + \left(\mu_1^{\sigma_1} - \mu_1 - \mu_1\frac{\partial \mu_1}{\Delta u_1}\right)^2\left(\frac{\partial^2 x}{\Delta u_1^2}\right)^2\\
&\quad +2\lambda\left(1+\frac{\partial \mu_1}{\Delta u_1}\right)\left(\frac{\partial h}{\Delta u_1}N + h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)\frac{\partial x}{\Delta u_1}^{\sigma_1}
\end{aligned}
$$

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$$+2\lambda\left(\mu_1^{\sigma_1}-\mu_1-\mu_1\frac{\partial\mu_1}{\Delta u_1}\right)\frac{\partial^2 x}{\Delta u_1^2}\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)+\lambda^2 c_{11}$$

$$=\left(1+\frac{\partial\mu_1}{\Delta u_1}\right)^2 F_{111}^{\sigma_1}+\widetilde{p}_{11}+2\lambda\left(1+\frac{\partial\mu_1}{\Delta u_1}\right)\left(\frac{\partial h}{\Delta u_1}N\frac{\partial x}{\Delta u_1}^{\sigma_1}-h^{\sigma_1}b_{11}\right)$$

$$+2\lambda\left(\mu_1^{\sigma_1}-\mu_1-\mu_1\frac{\partial\mu_1}{\Delta u_1}\right)\left(\frac{\partial h}{\Delta u_1}b_{11}+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\frac{\partial^2 x}{\Delta u_1^2}\right)+\lambda^2 c_{11}$$

$$=F_{111}+\lambda^2 c_{11}+\lambda q_{11}+p_{11},$$

i.e.,

$$\widetilde{F}_{111}=F_{111}+\lambda^2 c_{11}+\lambda q_{11}+p_{11},$$

where

$$\widetilde{p}_{111}=2\left(1+\frac{\partial\mu}{\Delta u_1}\right)\left(\mu_1^{\sigma_1}-\mu_1-\mu_1\frac{\partial\mu_1}{\Delta u_1}\right)\frac{\partial^2 x}{\Delta u_1^2}$$

$$+\left(\mu_1^{\sigma_1}-\mu_1-\mu_1\frac{\partial\mu_1}{\Delta u_1}\right)^2\left(\frac{\partial^2 x}{\partial u_1^2}\right)^2,$$

$$c_{11}=\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)^2,$$

$$q_{11}=2\left(1+\frac{\partial\mu}{\Delta u_1}\right)\left(\frac{\partial h}{\Delta u_1}N\frac{\partial x}{\Delta u_1}^{\sigma_1}-h^{\sigma_1}b_{11}\right)$$

$$+2\left(\mu_1^{\sigma_1}-\mu_1-\mu_1\frac{\partial\mu_1}{\Delta u_1}\right)\frac{\partial^2 x}{\Delta u_1^2}\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right),$$

$$p_{11}=\widetilde{p}_{111}+\mu_1\left(1+\frac{\partial\mu}{\Delta u_1}\right)^2 F_{111}^{\Delta_1}+\frac{\partial\mu}{\Delta u_1}\left(2+\frac{\partial\mu_1}{\Delta u_1}\right)F_{111}.$$

Next,

$$\widetilde{F}_{112}=\widetilde{F}_{121}$$

$$=\frac{\partial\widetilde{x}}{\Delta u_1}\frac{\partial\widetilde{x}}{\Delta u_2}^{\sigma_1}$$

$$=\left(\left(1+\frac{\partial\mu_1}{\Delta u_1}\right)\frac{\partial x}{\Delta u_1}^{\sigma_1}+\left(\mu_1^{\sigma_1}-\mu_1-\mu_1\frac{\partial\mu_1}{\Delta u_1}\right)\frac{\partial^2 x}{\Delta u_1^2}\right.$$

$$\left.+\lambda\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)\right)$$

$$
\begin{aligned}
&\times\left(\frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1}+\lambda\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1}+h^{\sigma_1\sigma_2}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right)\right)\\
&=\left(\left(1+\frac{\partial \mu_1}{\Delta u_1}\right)\left(\frac{\partial x}{\Delta u_1}^{\sigma_1}-\mu_1\frac{\partial^2 x}{\Delta u_1^2}\right)+\mu_1^{\sigma_1}\frac{\partial^2 x}{\Delta u_1^2}+\lambda\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)\right)\\
&\times\left(\frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1}+\lambda\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1}+h^{\sigma_1\sigma_2}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right)\right)\\
&=\left(1+\frac{\partial \mu_1}{\Delta u_1}\right)\left(\frac{\partial x}{\Delta u_1}^{\sigma_1}-\mu_1\frac{\partial^2 x}{\Delta u_1^2}\right)\frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1}+\mu_1^{\sigma_1}\frac{\partial^2 x}{\Delta u_1^2}\frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1}\\
&+\lambda\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)\frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1}\\
&+\lambda\left(\left(1+\frac{\partial \mu_1}{\Delta u_1}\right)\left(\frac{\partial x}{\Delta u_1}^{\sigma_1}-\mu_1\frac{\partial^2 x}{\Delta u_1^2}\right)+\mu_1^{\sigma_1}\frac{\partial^2 x}{\Delta u_1^2}\right)\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1}+h^{\sigma_1\sigma_2}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right)+\lambda^2 c_{12}\\
&=\left(1+\frac{\partial \mu_1}{\Delta u_1}\right)F_{112}^{\sigma_1}+\widetilde{p}_{12}+\lambda q_{12}+\lambda^2 c_{12}\\
&=\left(1+\frac{\partial \mu_1}{\Delta u_1}\right)\left(F_{112}+\mu_1\frac{\partial F_{112}}{\Delta u_1}\right)+\widetilde{p}_{12}+\lambda q_{12}+\lambda^2 c_{12}\\
&=F_{112}+\lambda^2 c_{12}+\lambda q_{12}+p_{12},
\end{aligned}
$$

where

$$
\begin{aligned}
c_{12}&=\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1}+h^{\sigma_1\sigma_2}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right),\\
\widetilde{p}_{12}&=\left(\mu_1^{\sigma_1}-\mu_1\left(1+\frac{\partial \mu_1}{\Delta u_1}\right)\right)\frac{\partial^2 x}{\Delta u_1^2}\frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1},\\
p_{12}&=F_{112}\frac{\partial \mu_1}{\Delta u_1}+\mu_1\frac{\partial \mu_1}{\Delta u_1}\frac{\partial F_{112}}{\Delta u_1}+\widetilde{p}_{12},\\
q_{12}&=\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)\frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1}\\
&+\left(\left(1+\frac{\partial \mu_1}{\Delta u_1}\right)\left(\frac{\partial x}{\Delta u_1}^{\sigma_1}-\mu_1\frac{\partial^2 x}{\Delta u_1^2}\right)+\mu_1^{\sigma_1}\frac{\partial^2 x}{\Delta u_1^2}\right)\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1}+h^{\sigma_1\sigma_2}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right)
\end{aligned}
$$

Moreover,

$$
\widetilde{F}_{122}=\left(\frac{\partial \widetilde{x}}{\Delta u_2}^{\sigma_1}\right)^2
$$

$$= \left(\frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1} + \lambda \left(\frac{\partial h}{\Delta u_2}^{\sigma_1} N^{\sigma_1} + h^{\sigma_1\sigma_2} \frac{\partial N}{\Delta u_2}^{\sigma_1} \right) \right)^2$$

$$= \left(\frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1} \right)^2 + 2\lambda \frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1} \left(\frac{\partial h}{\Delta u_2}^{\sigma_1} N + h^{\sigma_1\sigma_2} \frac{\partial N}{\Delta u_2}^{\sigma_1} \right) + \lambda^2 c_{22}$$

$$= F_{122}^{\sigma_1} + \lambda q_{22} + \lambda^2 c_{22}$$

$$= F_{122} + \mu_1 \frac{\partial F_{122}}{\Delta u_1} + \lambda q_{22} + \lambda^2 c_{22}$$

$$= F_{122} + \lambda^2 c_{22} + \lambda q_{22} + p_{22},$$

where

$$c_{22} = \left(\frac{\partial h}{\Delta u_2}^{\sigma_1} N^{\sigma_1} + h^{\sigma_1\sigma_2} \frac{\partial N}{\Delta u_2}^{\sigma_1} \right)^2,$$

$$q_{22} = 2 \frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1} \left(\frac{\partial h}{\Delta u_2}^{\sigma_1} N^{\sigma_1} + h^{\sigma_1\sigma_2} \frac{\partial N}{\Delta u_2}^{\sigma_1} \right),$$

$$p_{22} = \mu_1 \frac{\partial F_{122}}{\Delta u_1}.$$

Hence,

$$\begin{aligned}
\det(\widetilde{F}_{1ij}) &= \widetilde{F}_{11}\widetilde{F}_{22} - \widetilde{F}_{12}\widetilde{F}_{21} \\
&= (F_{111} + \lambda^2 c_{11} + \lambda q_{11} + p_{11})(F_{122} + \lambda^2 c_{22} + \lambda q_{22} + p_{22}) \\
&\quad -(F_{112} + \lambda^2 c_{12} + \lambda q_{12} + p_{12})(F_{121} + \lambda^2 c_{21} + \lambda q_{21} + p_{21}) \\
&= F_{111}F_{122} + \lambda^2 F_{111}c_{22} + \lambda F_{111}q_{22} + F_{111}p_{22} \\
&\quad +\lambda^2 c_{11}F_{122} + \lambda^4 c_{11}c_{22} + \lambda^3 c_{11}q_{22} + \lambda^2 c_{11}p_{22} \\
&\quad +\lambda q_{11}F_{122} + \lambda^3 q_{11}c_{22} + \lambda^2 q_{11}q_{22} + \lambda q_{11}p_{22} \\
&\quad +p_{11}F_{122} + \lambda^2 p_{11}c_{22} + \lambda p_{11}q_{22} + p_{11}p_{22} \\
&\quad -F_{112}F_{121} - \lambda^2 F_{112}c_{21} - \lambda F_{112}q_{21} - F_{112}p_{21}
\end{aligned}$$

$$
\begin{aligned}
&-\lambda^2 c_{12}F_{121} - \lambda^4 c_{12}c_{21} - \lambda^3 c_{12}q_{21} - \lambda^2 c_{12}p_{21} \\
&-\lambda q_{12}F_{121} - \lambda^3 q_{12}c_{21} - \lambda^2 q_{12}q_{21} - \lambda q_{12}p_{21} \\
&+p_{12}F_{121} + \lambda^2 p_{12}c_{21} + \lambda p_{12}q_{21} + p_{12}p_{21} \\
=\;& \lambda^4(c_{11}c_{22} - c_{12}c_{21}) \\
&+\lambda^3\Big(c_{11}q_{22} + q_{11}c_{22} - c_{12}q_{21} - q_{12}c_{21}\Big) \\
&+\lambda^2\Big(F_{111}c_{22} + c_{11}F_{122} + c_{11}p_{22} + q_{11}q_{22} + p_{11}c_{22} - F_{112}c_{21} \\
&-c_{12}F_{121} - c_{12}p_{21} - q_{12}q_{21} + p_{12}c_{21}\Big) \\
&+\lambda\Big(F_{111}q_{22} + q_{11}F_{122} + q_{11}p_{22} + p_{11}q_{22} - F_{112}q_{21} \\
&-q_{12}F_{121} - q_{12}p_{21} + p_{12}q_{21}\Big) \\
&+F_{111}F_{122} - F_{112}F_{121} + F_{111}p_{22} + p_{11}F_{122} + p_{11}p_{22} \\
&-F_{112}p_{21} + p_{12}F_{121} + p_{12}p_{21} \\
=\;& \lambda^4(c_{11}c_{22} - c_{12}c_{21}) \\
&+\lambda^3\Big(c_{11}q_{22} + q_{11}c_{22} - c_{12}q_{21} - q_{12}c_{21}\Big) \\
&+\lambda^2\Big(F_{111}c_{22} + c_{11}F_{122} + c_{11}p_{22} + q_{11}q_{22} + p_{11}c_{22} - F_{112}c_{21} \\
&-c_{12}F_{121} - c_{12}p_{21} - q_{12}q_{21} + p_{12}c_{21}\Big) \\
&+\lambda\Big(F_{111}q_{22} + q_{11}F_{122} + q_{11}p_{22} + p_{11}q_{22} - F_{112}q_{21} \\
&-q_{12}F_{121} - q_{12}p_{21} + p_{12}q_{21}\Big)
\end{aligned}
$$

$$+\det(F_{1ij})+F_{111}p_{22}+p_{11}F_{122}+p_{11}p_{22}$$

$$-F_{112}p_{21}+p_{12}F_{121}+p_{12}p_{21}.$$

Observe that

$$\begin{aligned}
&c_{11}c_{22}-c_{12}c_{21}\\
&=\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)^2\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1}+h^{\sigma_1\sigma_2}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right)^2\\
&\quad-\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)^2\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1}+h^{\sigma_1\sigma_2}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right)^2\\
&=0
\end{aligned}$$

and

$$\begin{aligned}
&c_{11}q_{22}+q_{11}c_{22}-c_{12}q_{21}-q_{12}c_{21}\\
&=2\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)^2\frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1}\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1}+h^{\sigma_1\sigma_2}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right)\\
&\quad+2\left(\left(1+\frac{\partial \mu}{\Delta u_1}\right)\left(\frac{\partial h}{\Delta u_1}N\frac{\partial x}{\Delta u_1}^{\sigma_1}-h^{\sigma_1}b_{11}\right)\right.\\
&\quad\left.+\left(\mu_1^{\sigma_1}-\mu_1-\mu_1\frac{\partial \mu_1}{\Delta u_1}\right)\frac{\partial^2 x}{\Delta u_1^2}\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)\right)\\
&\quad\times\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1}+h^{\sigma_1\sigma_2}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right)^2\\
&\quad-2\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1}+h^{\sigma_1\sigma_2}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right)\\
&\quad\times\left(\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)\frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1}\right.\\
&\quad\left.+\left(\left(1+\frac{\partial \mu_1}{\Delta u_1}\right)\left(\frac{\partial x}{\Delta u_1}^{\sigma_1}-\mu_1\frac{\partial^2 x}{\Delta u_1^2}\right)+\mu_1^{\sigma_1}\frac{\partial^2 x}{\Delta u_1^2}\right)\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1}+h^{\sigma_1\sigma_2}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right)\right)\\
&=2\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)^2\frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1}\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1}+h^{\sigma_1\sigma_2}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right)\\
&\quad+2\left(1+\frac{\partial \mu}{\Delta u_1}\right)\left(\frac{\partial h}{\Delta u_1}N\frac{\partial x}{\Delta u_1}^{\sigma_1}-h^{\sigma_1}b_{11}\right)\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1}+h^{\sigma_1\sigma_2}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right)^2
\end{aligned}$$

$$+2\left(\mu_1^{\sigma_1}-\mu_1-\mu_1\frac{\partial\mu_1}{\Delta u_1}\right)\frac{\partial^2 x}{\Delta u_1^2}\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1}+h^{\sigma_1\sigma_2}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right)^2$$

$$-2\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)^2\frac{\partial x}{\Delta u_2}^{\sigma_1\sigma_1}\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1}+h^{\sigma_1\sigma_2}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right)$$

$$-2\left(1+\frac{\partial\mu}{\Delta u_1}\right)\left(\frac{\partial h}{\Delta u_1}N\frac{\partial x}{\Delta u_1}^{\sigma_1}-h^{\sigma_1}b_{11}\right)\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1}+h^{\sigma_1\sigma_2}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right)^2$$

$$-2\left(\mu_1^{\sigma_1}-\mu_1-\mu_1\frac{\partial\mu_1}{\Delta u_1}\right)\frac{\partial^2 x}{\Delta u_1^2}\left(\frac{\partial h}{\Delta u_1}N+h^{\sigma_1}\frac{\partial N}{\Delta u_1}\right)\left(\frac{\partial h}{\Delta u_2}^{\sigma_1}N^{\sigma_1}+h^{\sigma_1\sigma_2}\frac{\partial N}{\Delta u_2}^{\sigma_1}\right)^2$$

$$=0.$$

Let

$$\begin{aligned}
a_0 &= \det(F_{1ij})+F_{111}p_{22}+p_{11}F_{122}+p_{11}p_{22}\\
&\quad -F_{112}p_{21}+p_{12}F_{121}+p_{12}p_{21},\\
a_1 &= F_{111}q_{22}+q_{11}F_{122}+q_{11}p_{22}+p_{11}q_{22}-F_{112}q_{21}\\
&\quad -q_{12}F_{121}-q_{12}p_{21}+p_{12}q_{21},\\
a_2 &= F_{111}c_{22}+c_{11}F_{122}+c_{11}p_{22}+q_{11}q_{22}+p_{11}c_{22}-F_{112}c_{21}\\
&\quad -c_{12}F_{121}-c_{12}p_{21}-q_{12}q_{21}+p_{12}c_{21}.
\end{aligned}$$

Therefore

$$\det(\widetilde{F}_{1ij})=a_2\lambda^2+a_1\lambda+a_0.$$

Since S is σ_1-regular, we conclude that a_0 has a positive minimum on $\widetilde{Q}$ and using that $a_1,a_2\in\mathscr{C}(U)$, we have that there exists $\varepsilon>0$ such that $\det(\widetilde{F}_{1ij})>0$ for $|\lambda|<\varepsilon$ and $u\in\overline{Q}$. Thus, for $|\lambda|<\varepsilon$, the surfaces Σ_λ defined by restricting $\widetilde{x}(u)$ to Q, are σ_1-regular surfaces. Note that

$$\left|\sqrt{\det\widetilde{F}_{1ij}}-\left(\sqrt{a_0}+\frac{a_1}{2\sqrt{a_0}}\lambda\right)\right|<M\lambda^2,\quad u\in Q, \tag{2.14}$$

for some positive constant M.

Now, we consider a surface in nonparametric form

$$x_3=f(x_1,x_2),\quad (x_1,x_2)\in U,$$

where $f \in \mathscr{C}^2(U)$. Introduce the notations

$$p^{\sigma_1} = \frac{\partial f}{\Delta x_1},$$

$$q = \frac{\partial f}{\Delta x_2},$$

$$r = \frac{\partial^2 f}{\Delta x_1^2},$$

$$s = \frac{\partial}{\Delta x_1}\left(\frac{\partial f}{\Delta x_2}^{\sigma_1}\right),$$

$$t = \frac{\partial}{\Delta x_2}\left(\frac{\partial f}{\Delta x_2}^{\sigma_1}\right),$$

$$l = \frac{\partial^2 f^{\sigma_1}}{\Delta x_2 \Delta x_1}.$$

Then, the equation (2.10) for a minimal surface takes the form

$$\left(1+(q^{\sigma_1})^2\right)\frac{\partial p^{\sigma_1}}{\Delta x_1} + \left(1+(p^{\sigma_1})^2\right)\left(\frac{\partial q^{\sigma_1}}{\Delta x_2}\right)^{\sigma_1}$$

$$-p^{\sigma_1} q^{\sigma_1}\frac{\partial q^{\sigma_1}}{\Delta x_1} - p^{\sigma_1} q^{\sigma_1}\frac{\partial}{\Delta x_2}(p^{\sigma_1\sigma_1}) = 0$$

or

$$\left(1+(q^{\sigma_1})^2\right) r + \left(1+(p^{\sigma_1})^2\right) t = p^{\sigma_1} q^{\sigma_1} s + p^{\sigma_1} q^{\sigma_1} l,$$

or

$$\left(1+(q^{\sigma_1})^2\right) r - p^{\sigma_1} q^{\sigma_1}(s+l) + \left(1+(p^{\sigma_1})^2\right) t = 0.$$

Next,

$$F_{111} = 1+(p^{\sigma_1})^2,$$

$$F_{112} = p^{\sigma_1} q^{\sigma_1},$$

$$F_{122} = 1+(q^{\sigma_1})^2$$

and

$$\det(F_{1ij}) = F_{111}F_{122} - F_{112}^2$$

$$= \left(1+(p^{\sigma_1})^2\right)\left(1+(q^{\sigma_1})^2\right) - (p^{\sigma_1} q^{\sigma_1})^2$$

$$= 1 + (p^{\sigma_1})^2 + (q^{\sigma_1})^2.$$

Let

$$W^{\sigma_1} = \sqrt{\det(F_{1ij})}.$$

Then

$$(W^{\sigma_1})^2 = \det(F_{1ij}).$$

Set

$$\widetilde{f} = f + \lambda h, \quad \lambda \in \mathbb{R},$$

where $h \in \mathscr{C}^2(U)$ is an arbitrarily chosen function. Then

$$\widetilde{p}^{\sigma_1} = p^{\sigma_1} + \lambda \frac{\partial h}{\Delta x_1},$$

$$\widetilde{q}^{\sigma_1} = q^{\sigma_1} + \lambda \frac{\partial h}{\Delta x_2}^{\sigma_1},$$

whereupon

$$(\widetilde{W}^{\sigma_1})^2 = 1 + (\widetilde{p}^{\sigma_1})^2 + (\widetilde{q}^{\sigma_1})^2$$

$$= 1 + \left(p^{\sigma_1} + \lambda \frac{\partial h}{\Delta x_1}\right)^2 + \left(q^{\sigma_1} + \lambda \frac{\partial h}{\Delta x_2}^{\sigma_1}\right)^2$$

$$= 1 + (p^{\sigma_1})^2 + 2\lambda p^{\sigma_1} \frac{\partial h}{\Delta x_1} + \lambda^2 \left(\frac{\partial h}{\Delta x_1}\right)^2 + (q^{\sigma_1})^2 + 2\lambda q^{\sigma_1} \frac{\partial h}{\Delta x_2}^{\sigma_1} + \lambda^2 \left(\frac{\partial h}{\Delta x_2}^{\sigma_1}\right)^2$$

$$= (W^{\sigma_1})^2 + 2\lambda \left(p^{\sigma_1} \frac{\partial h}{\Delta x_1} + q^{\sigma_1} \frac{\partial h}{\Delta x_2}^{\sigma_1}\right) + \lambda^2 \left(\left(\frac{\partial h}{\Delta x_1}\right)^2 + \left(\frac{\partial h}{\Delta x_2}^{\sigma_1}\right)^2\right)$$

$$= (W^{\sigma_1})^2 + 2\lambda X + \lambda^2 Y,$$

where

$$X = p^{\sigma_1} \frac{\partial h}{\Delta x_1} + q^{\sigma_1} \frac{\partial h}{\Delta x_2}^{\sigma_1},$$

$$Y = \left(\frac{\partial h}{\Delta x_1}\right)^2 + \left(\frac{\partial h}{\Delta x_2}^{\sigma_1}\right)^2.$$

Thus,

$$\widetilde{W}^{\sigma_1} = W^{\sigma_1} + \lambda \frac{X}{W^{\sigma_1}} + \lambda^2 Z,$$

where Z is a continuous function of x_1 and x_2.

Now, we consider a closed curve Γ in the domain of the definition of the function f and let Δ be the region bounded by Γ. If the surface $x_3 = f(x_1,x_2)$ over Q minimizes the area along all surfaces with the same boundary, then for any choice of h such that $h = 0$ and h^{σ_1} =on Γ, we have

$$\int\int_Q \widetilde{W}^{\sigma_1}\Delta x_1 \Delta x_2 \geq \int\int_Q W^{\sigma_1}\Delta x_1 \Delta x_2,$$

which is possible if

$$\int\int_\Delta \frac{X}{W^{\sigma_1}}\Delta x_1 \Delta x_2 \geq 0.$$

Substituting in the above expression X, then integrating by parts and using the $h = 0$ and $h^{\sigma_1} = 0$ on Γ, we find

$$\begin{aligned} 0 &= \int\int_Q \left(\frac{p^{\sigma_1}}{W^{\sigma_1}}\frac{\partial h}{\Delta x_1} + \frac{q^{\sigma_1}}{W^{\sigma_1}}\frac{\partial h}{\Delta x_2}^{\sigma_1}\right)\Delta x_1 \Delta x_2 \\ &= \int\int_Q \left(-\frac{\partial}{\Delta_1 x_1}\left(\frac{p^{\sigma_1}}{W^{\sigma_1}}\right)h^{\sigma_1} - \frac{\partial}{\Delta x_2}\left(\frac{q^{\sigma_1}}{W^{\sigma_1}}\right)h^{\sigma_1}\right)\Delta x_1 \Delta x_2 \\ &= -\int\int_Q \left(\left(\frac{\partial}{\Delta x_1}\left(\frac{p^{\sigma_1}}{W^{\sigma_1}}\right) + \frac{\partial}{\Delta x_2}\left(\frac{q^{\sigma_1}}{W^{\sigma_1}}\right)\right)\right)h^{\sigma_1}\Delta x_1 \Delta x_2, \end{aligned}$$

whereupon

$$\frac{\partial}{\Delta x_1}\left(\frac{p^{\sigma_1}}{W^{\sigma_1}}\right) + \frac{\partial}{\Delta x_2}\left(\frac{q^{\sigma_1}}{W^{\sigma_1}}\right) = 0. \tag{2.15}$$

2.4 σ_1-Isothermal Parameters

Definition 2.23. Parameters u_1 and u_2 for which

$$F_{111} = F_{122},$$

$$F_{112} = 0$$

are said to be σ_1-isothermal parameters. In other words, the parameters u_1 and u_2 are said to be σ_1-isothermal of

$$F_{1ij} = \lambda^2 \delta_{ij}, \quad i,j \in \{1,2\},$$

where $\lambda = \lambda(u) > 0$, $u \in U$, and δ_{ij}, $i,j \in \{1,2\}$, are the Kronecker coefficients.

Suppose that u_1 and u_2 are σ_1-isothermal parameters. Then

$$\det(F_{1ij}) = \lambda^4$$

and

$$H(N) = \frac{\lambda^2 b_{11}(N) + \lambda^2 b_{22}(N)}{2\lambda^4}$$

$$= \frac{b_{11}(N) + b_{22}(N)}{2\lambda^2}.$$

Definition 2.24. σ_1-Laplacian is defined by

$$\Delta^{\sigma_1} x = \frac{\partial^2 x}{\Delta u_1^2} + \frac{\partial}{\Delta u_2}\left(\frac{\partial x}{\Delta u_2}^{\sigma_1}\right)^{\sigma_1}$$

for any $x \in \mathscr{C}^2(U)$.

Theorem 2.5. *Let S be a σ_1-regular surface defined by $x = x(u)$, $u \in U$, $x \in \mathscr{C}^2(U)$, where u_1 and u_2 are σ_1-isothermal parameters. Then*

$$\Delta^{\sigma_1} x = -2\lambda^2 H. \tag{2.16}$$

Proof. We have

$$\Delta^{\sigma_1} xN = \left(\frac{\partial^2 x}{\Delta u_1^2} + \frac{\partial}{\Delta u_2}\left(\frac{\partial x}{\Delta u_2}^{\sigma_1}\right)^{\sigma_1}\right)N$$

$$= \frac{\partial^2 x}{\Delta u_1^2}N + \frac{\partial}{\Delta u_2}\left(\frac{\partial x}{\Delta u_2}^{\sigma_1}\right)^{\sigma_1}N$$

$$= -b_{11}(N) - b_{22}(N)$$

$$= -2\lambda^2 H(N)$$

$$= -2\lambda^2 HN,$$

whereupon we get (2.16). This completes the proof.

Definition 2.25. A function $x \in \mathscr{C}^2(U)$ is said to be σ_1-harmonic if

$$\Delta^{\sigma_1} x = 0 \quad \text{on} \quad U.$$

Corollary 2.3. *Let $x = x(u)$, $u \in U$, $x \in \mathscr{C}^2(U)$, define σ_1-regular surface S with σ_1-isothermal parameters. Necessary and sufficient conditions that the coordinate functions $x_k(u_1, u_2)$, $k \in \{1, 2, 3\}$, be σ_1-harmonic is that S is a minimal surface.*

Proof. 1. Let the coordinate functions x_k, $k \in \{1, 2, 3\}$, be σ_1-harmonic. Then, using (2.16), we find $H = 0$, i.e., S is a minimal surface.

2. Let S be a minimal surface. Then $H = 0$ and using (2.16), we find

$$\Delta^{\sigma_1} x_k = 0, \quad k \in \{1,2,3\}, \quad \text{on} \quad U,$$

i.e., the coordinate functions x_k, $k \in \{1,2,3\}$, are σ_1-harmonic. This completes the proof.

We introduce the following notation. Suppose that S is a surface defined $x = x(u)$, $u \in U$, $x \in \mathscr{C}^2(U)$. Consider the complex valued function

$$\phi_k(\zeta) = \frac{\partial x_k}{\Delta u_1} - i \frac{\partial x_k}{\Delta u_2}^{\sigma_1},$$

$$\zeta = u_1 + iu_2, \quad k \in \{1,2,3\}.$$

Then

$$\begin{aligned}
\sum_{k=1}^{3} (\phi_k(\zeta))^2 &= \sum_{k=1}^{3} \left(\frac{\partial x_k}{\Delta u_1} - i \frac{\partial x_k}{\Delta u_2}^{\sigma_1} \right)^2 \\
&= \sum_{k=1}^{3} \left(\frac{\partial x_k}{\Delta u_1} \right)^2 - 2i \sum_{k=1}^{3} \frac{\partial x_k}{\Delta u_1} \frac{\partial x_k}{\Delta u_2}^{\sigma_1} - \sum_{k=1}^{3} \left(\frac{\partial x_k}{\Delta u_2}^{\sigma_1} \right)^2 \\
&= \left| \frac{\partial x}{\Delta u_1} \right|^2 - 2i \frac{\partial x}{\Delta u_1} \frac{\partial x}{\Delta u_2}^{\sigma_1} - \left| \frac{\partial x}{\Delta u_2}^{\sigma_1} \right|^2 \\
&= F_{111} - F_{122} - 2iF_{112}.
\end{aligned}$$

Next,

$$|\phi_k(\zeta)|^2 = \left(\frac{\partial x_k}{\Delta u_1} \right)^2 + \left(\frac{\partial x_k}{\Delta u_2}^{\sigma_1} \right)^2, \quad k \in \{1,2,3\},$$

$$\begin{aligned}
\sum_{k=1}^{3} |\phi_k(\zeta)|^2 &= \sum_{k=1}^{3} \left(\left(\frac{\partial x_k}{\Delta u_1} \right)^2 + \left(\frac{\partial x_k}{\Delta u_2}^{\sigma_1} \right)^2 \right) \\
&= \sum_{k=1}^{3} \left(\frac{\partial x_k}{\Delta u_1} \right)^2 + \sum_{k=1}^{3} \left(\frac{\partial x_k}{\Delta u_2}^{\sigma_1} \right)^2 \\
&= F_{111} + F_{122}.
\end{aligned}$$

Definition 2.26. We will say that $\phi_k(\zeta)$ is analytic in ζ if x_k, $k \in \{1,2,3\}$, are σ_1-harmonic in u_1 and u_2 in the whole U.

Theorem 2.6. *The parameters u_1 and u_2 are σ_1-isothermal parameters if and only if*

$$\sum_{k=1}^{3} (\phi_k(\zeta))^2 = 0. \tag{2.17}$$

Proof. 1. Let u_1 and u_2 be σ_1-isothermal parameters. Hence, by the definition for σ_1-isothermal parameters, we get

$$F_{111} = F_{122},$$

$$F_{112} = 0.$$

Hence,

$$\sum_{k=1}^{3} (\phi_k(\zeta))^2 = F_{111} - F_{122} - 2iF_{112}$$

$$= 0.$$

2. Let (2.17) holds. Then

$$F_{111} - F_{122} - 2iF_{112} = 0.$$

Hence,

$$F_{111} = F_{122} \quad \text{and} \quad F_{112} = 0.$$

This completes the proof.

Theorem 2.7. *Let u_1 and u_2 be σ_1-isothermal parameters. Then S is a σ_1-regular surface if and only if*

$$\sum_{k=1}^{3} |\phi_k(\zeta)|^2 \neq 0. \tag{2.18}$$

Proof. 1. Let S be a σ_1-regular surface. Then

$$\sum_{k=1}^{3} \left(\frac{\partial x_k}{\Delta u_1}\right)^2 + \sum_{k=1}^{3} \left(\frac{\partial x_k}{\Delta u_2}^{\sigma_1}\right)^2 \neq 0, \tag{2.19}$$

or (2.18) holds.

2. Let (2.18) holds. Then (2.19) holds and the surface S is a σ_1-regular surface. This completes the proof.

Theorem 2.8. *Let $x = x(u)$, $u \in U$, $x \in \mathscr{C}^2(U)$, be a σ_1-regular minimal surface with u_1 and u_2 σ_1-isothermal parameters. Then the functions $\phi_k(\zeta)$, $k \in \{1,2,3\}$, are analytic functions and they satisfy* (2.17) *and* (2.18)*. Conversely, let $\phi_k(\zeta)$, $k \in \{1,2,3\}$, be analytic functions which satisfy* (2.17) *and* (2.18) *in a simply-connected domain U. Then there exists a σ_1-regular minimal surface $x = x(u)$, $u \in U$, $x \in \mathscr{C}^2(U)$, such that*

$$\phi_k(\zeta) = \frac{\partial x_k}{\Delta u_1} - i\frac{\partial x_k}{\Delta u_2}^{\sigma_1}, \quad k \in \{1,2,3\}. \tag{2.20}$$

Proof. 1. Let $x = x(u)$, $u \in U$, $x \in \mathscr{C}^2(U)$, be a σ_1-regular minimal surface with u_1 and u_2 σ_1-isothermal parameters. Then, applying Theorem 2.6, it follows that (2.17) holds. By Theorem 2.7, it follows that (2.18) holds.

2. Let $\phi_k(\zeta)$, $k \in \{1,2,3\}$, be analytic functions of ζ that satisfy (2.17) and (2.18). Define

$$x_k = \operatorname{Re}\int \phi_k(\zeta)\Delta\zeta, \quad k \in \{1,2,3\}.$$

Then x_k, $k \in \{1,2,3\}$, are σ_1-harmonic functions and

$$\phi_k(\zeta) = \frac{\partial x_k}{\Delta u_1} - i\frac{\partial x_k}{\Delta u_2}^{\sigma_1}, \quad k \in \{1,2,3\}.$$

Since x_k, $k \in \{1,2,3\}$, are σ_1-harmonic functions, we get

$$\frac{\partial^2 x_k}{\Delta u_1^2} + \frac{\partial}{\Delta u_2}\left(\frac{\partial x_k}{\Delta u_2}^{\sigma_1}\right)^{\sigma_1} = 0, \quad k \in \{1,2,3\}.$$

By Corollary 2.3, it follows that $x = x(u)$, $u \in U$, is a σ_1-regular minimal surface. This completes the proof.

Theorem 2.9. *Let S be a minimal surface. Then any σ_1-regular point (x_1,x_2) of S, $(x_1,x_2) \in U \subset \mathbb{T}_1 \times \mathbb{T}_2$, has a neighbourhood in which there exists a parameterization $(\xi_1,\xi_2) \in \mathbb{T}_{(1)} \times \mathbb{T}_{(2)}$ of S so that*

$$\left(\frac{\partial x}{\Delta \xi_1}\right)^2 = \left(\frac{\partial x}{\Delta \xi_2}\right)^2 \quad \text{and} \quad \frac{\partial x}{\Delta \xi_1}\frac{\partial x}{\Delta \xi_2} = 0. \tag{2.21}$$

Definition 2.27. Let (x_1,x_2) and $\xi_1,\xi_2)$ be as in Theorem 2.9. Then, we say that ξ_1,ξ_2 are in terms of σ_1-isothermal parameters.

Proof. By Theorem 2.2, it follows that we can find a neighbourhood of the σ_1-regular point in which S may be represented in a nonparametric form. Then, the equation (2.15) is satisfied in some disc

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 < R^2.$$

By the equation (2.15), it follows that there is a σ_1-completely delta differentiable function F such that

$$\frac{\partial F}{\Delta x_2} = \frac{p^{\sigma_1}}{W^{\sigma_1}}$$

$$\frac{\partial F}{\Delta x_1} = -\frac{q}{W}.$$

Now, we take a σ_1-completely delta differentiable function G so that

$$
\begin{aligned}
\frac{\partial G}{\Delta x_1} &= -\frac{\partial F}{\Delta x_2}^{\sigma_1} \\
\frac{\partial G}{\Delta x_2}^{\sigma_1} &= \frac{\partial F}{\Delta x_1}.
\end{aligned}
\tag{2.22}
$$

Set

$$
\begin{aligned}
\xi_1 &= x_1 + F(x_1, x_2), \\
\xi_2 &= x_2 + G(x_1, x_2).
\end{aligned}
\tag{2.23}
$$

Then, after we differentiate the first equation of (2.23) with respect to ξ_1 and ξ_2, we get

$$
\begin{aligned}
1 &= \frac{\partial x_1}{\Delta \xi_1} + \frac{\partial F}{\Delta x_1}\frac{\partial x_1}{\Delta \xi_1} + \frac{\partial F}{\Delta x_2}^{\sigma_1}\frac{\partial x_2}{\Delta \xi_1} \\
&= \left(1 + \frac{\partial F}{\Delta x_1}\right)\frac{\partial x_1}{\Delta \xi_1} + \frac{\partial F}{\Delta x_2}^{\sigma_1}\frac{\partial x_2}{\Delta \xi_1}, \\
0 &= \frac{\partial x_1}{\Delta \xi_2} + \frac{\partial F}{\Delta x_1}\frac{\partial x_1}{\Delta \xi_2} + \frac{\partial F}{\Delta x_2}^{\sigma_1}\frac{\partial x_2}{\Delta \xi_2} \\
&= \left(1 + \frac{\partial F}{\Delta x_1}\right)\frac{\partial x_1}{\Delta \xi_2} + \frac{\partial F}{\Delta x_2}^{\sigma_1}\frac{\partial x_2}{\Delta \xi_2},
\end{aligned}
$$

i.e., we get the system

$$
\begin{aligned}
1 &= \left(1 + \frac{\partial F}{\Delta x_1}\right)\frac{\partial x_1}{\Delta \xi_1} + \frac{\partial F}{\Delta x_2}^{\sigma_1}\frac{\partial x_2}{\Delta \xi_1} \\
0 &= \left(1 + \frac{\partial F}{\Delta x_1}\right)\frac{\partial x_1}{\Delta \xi_2} + \frac{\partial F}{\Delta x_2}^{\sigma_1}\frac{\partial x_2}{\Delta \xi_2}.
\end{aligned}
\tag{2.24}
$$

Now, we differentiate the second equation of (2.23) with respect to ξ_1 and ξ_2 and we find

$$
\begin{aligned}
0 &= \frac{\partial x_2}{\Delta \xi_1} + \frac{\partial G}{\Delta x_1}\frac{\partial x_1}{\Delta \xi_1} + \frac{\partial G}{\Delta x_2}^{\sigma_1}\frac{\partial x_2}{\Delta \xi_1} \\
&= \left(1 + \frac{\partial G}{\Delta x_2}^{\sigma_1}\right)\frac{\partial x_2}{\Delta \xi_1} + \frac{\partial G}{\Delta x_1}\frac{\partial x_1}{\Delta \xi_1}
\end{aligned}
$$

and

$$
1 = \frac{\partial x_2}{\Delta \xi_2} + \frac{\partial G}{\Delta x_1}\frac{\partial x_1}{\Delta \xi_2} + \frac{\partial G}{\Delta x_2}^{\sigma_1}\frac{\partial x_2}{\Delta \xi_2}
$$

$$0 = \left(1 + \frac{\partial G}{\Delta x_2}^{\sigma_1}\right)\frac{\partial x_2}{\Delta \xi_2} + \frac{\partial G}{\Delta x_1}\frac{\partial x_1}{\Delta \xi_2}.$$

Thus, we get the system

$$\begin{aligned} 0 &= \left(1 + \frac{\partial G}{\Delta x_2}^{\sigma_1}\right)\frac{\partial x_2}{\Delta \xi_1} + \frac{\partial G}{\Delta x_1}\frac{\partial x_1}{\Delta \xi_1} \\ 1 &= \left(1 + \frac{\partial G}{\Delta x_2}^{\sigma_1}\right)\frac{\partial x_2}{\Delta \xi_2} + \frac{\partial G}{\Delta x_1}\frac{\partial x_1}{\Delta \xi_2}. \end{aligned} \tag{2.25}$$

By the first equations of (2.24) and (2.25), we obtain the system

$$\begin{aligned} 1 &= \left(1 + \frac{\partial F}{\Delta x_1}\right)\frac{\partial x_1}{\Delta \xi_1} + \frac{\partial F}{\Delta x_2}^{\sigma_1}\frac{\partial x_2}{\Delta \xi_1} \\ 0 &= \left(1 + \frac{\partial G}{\Delta x_2}^{\sigma_1}\right)\frac{\partial x_2}{\Delta \xi_1} + \frac{\partial G}{\Delta x_1}\frac{\partial x_1}{\Delta \xi_1}, \end{aligned}$$

whereupon

$$\begin{aligned} \frac{\partial x_1}{\Delta \xi_1} &= \frac{1 + \frac{\partial G}{\Delta x_2}^{\sigma_1}}{1 + \frac{\partial F}{\Delta x_1} + \frac{\partial G}{\Delta x_2}^{\sigma_1} + \frac{\partial F}{\Delta x_1}\frac{\partial G}{\Delta x_2}^{\sigma_1} - \frac{\partial F}{\Delta x_2}^{\sigma_1}\frac{\partial G}{\Delta x_1}} \\ \frac{\partial x_2}{\Delta \xi_1} &= -\frac{\frac{\partial G}{\Delta x_1}}{1 + \frac{\partial F}{\Delta x_1} + \frac{\partial G}{\Delta x_2}^{\sigma_1} + \frac{\partial F}{\Delta x_1}\frac{\partial G}{\Delta x_2}^{\sigma_1} - \frac{\partial F}{\Delta x_2}^{\sigma_1}\frac{\partial G}{\Delta x_1}}. \end{aligned}$$

By the second equations of (2.24) and (2.25), we get

$$\begin{aligned} 0 &= \left(1 + \frac{\partial F}{\Delta x_1}\right)\frac{\partial x_1}{\Delta \xi_2} + \frac{\partial F}{\Delta x_2}^{\sigma_1}\frac{\partial x_2}{\Delta \xi_2} \\ 1 &= \left(1 + \frac{\partial G}{\Delta x_2}^{\sigma_1}\right)\frac{\partial x_2}{\Delta \xi_2} + \frac{\partial G}{\Delta x_1}\frac{\partial x_1}{\Delta \xi_2}, \end{aligned}$$

from where

$$\begin{aligned} \frac{\partial x_1}{\Delta \xi_2} &= -\frac{\frac{\partial F}{\Delta x_2}^{\sigma_1}}{1 + \frac{\partial F}{\Delta x_1} + \frac{\partial G}{\Delta x_2}^{\sigma_1} + \frac{\partial F}{\Delta x_1}\frac{\partial G}{\Delta x_2}^{\sigma_1} - \frac{\partial F}{\Delta x_2}^{\sigma_1}\frac{\partial G}{\Delta x_1}} \\ \frac{\partial x_2}{\Delta \xi_2} &= \frac{1 + \frac{\partial F}{\Delta x_1}}{1 + \frac{\partial F}{\Delta x_1} + \frac{\partial G}{\Delta x_2}^{\sigma_1} + \frac{\partial F}{\Delta x_1}\frac{\partial G}{\Delta x_2}^{\sigma_1} - \frac{\partial F}{\Delta x_2}^{\sigma_1}\frac{\partial G}{\Delta x_1}}. \end{aligned}$$

By (2.22), we find

$$\frac{\partial x_1}{\Delta \xi_1} = \frac{1+\frac{\partial F}{\Delta x_1}}{\left(1+\frac{\partial F}{\Delta x_1}\right)^2 + \left(\frac{\partial F}{\Delta x_2}^{\sigma_1}\right)^2}$$

$$\frac{\partial x_2}{\Delta \xi_1} = \frac{\frac{\partial F}{\Delta x_2}^{\sigma_1}}{\left(1+\frac{\partial F}{\Delta x_1}\right)^2 + \left(\frac{\partial F}{\Delta x_2}^{\sigma_1}\right)^2}$$

and

$$\frac{\partial x_1}{\Delta \xi_2} = -\frac{\frac{\partial F}{\Delta x_2}^{\sigma_1}}{\left(1+\frac{\partial F}{\Delta x_1}\right)^2 + \left(\frac{\partial F}{\Delta x_2}^{\sigma_1}\right)^2}$$

$$\frac{\partial x_2}{\Delta \xi_2} = \frac{1+\frac{\partial F}{\Delta x_1}}{\left(1+\frac{\partial F}{\Delta x_1}\right)^2 + \left(\frac{\partial F}{\Delta x_2}^{\sigma_1}\right)^2}.$$

Then

$$\begin{aligned}
\left(\frac{\partial x_1}{\Delta \xi_1}\right)^2 + \left(\frac{\partial x_2}{\Delta \xi_1}\right)^2 &= \frac{\left(1+\frac{\partial F}{\Delta x_1}\right)^2 + \left(\frac{\partial F}{\Delta x_2}^{\sigma_1}\right)^2}{\left(\left(1+\frac{\partial F}{\Delta x_1}\right)^2 + \left(\frac{\partial F}{\Delta x_2}^{\sigma_1}\right)^2\right)^2} \\
&= \frac{1}{\left(1+\frac{\partial F}{\Delta x_1}\right)^2 + \left(\frac{\partial F}{\Delta x_2}^{\sigma_1}\right)^2}, \\
\left(\frac{\partial x_1}{\Delta \xi_2}\right)^2 + \left(\frac{\partial x_2}{\Delta \xi_2}\right)^2 &= \frac{\left(1+\frac{\partial F}{\Delta x_1}\right)^2 + \left(\frac{\partial F}{\Delta x_2}^{\sigma_1}\right)^2}{\left(\left(1+\frac{\partial F}{\Delta x_1}\right)^2 + \left(\frac{\partial F}{\Delta x_2}^{\sigma_1}\right)^2\right)^2} \\
&= \frac{1}{\left(1+\frac{\partial F}{\Delta x_1}\right)^2 + \left(\frac{\partial F}{\Delta x_2}^{\sigma_1}\right)^2}, \\
\frac{\partial x_1}{\Delta \xi_1}\frac{\partial x_1}{\Delta \xi_2} + \frac{\partial x_2}{\Delta \xi_1}\frac{\partial x_2}{\Delta \xi_2} &= \frac{1}{\left(\left(1+\frac{\partial F}{\Delta x_1}\right)^2 + \left(\frac{\partial F}{\Delta x_2}^{\sigma_1}\right)^2\right)^2} \\
&\quad \times \left(-\left(1+\frac{\partial F}{\Delta x_1}\right)\frac{\partial F}{\Delta x_2}^{\sigma_1} + \frac{\partial F}{\Delta x_2}^{\sigma_1}\left(1+\frac{\partial F}{\Delta x_1}\right)\right) \\
&= 0,
\end{aligned}$$

i.e., ξ_1 and ξ_2 are in terms of σ_1-isothermal parameters. This completes the proof.

Theorem 2.10. *Let a surface S be defined by $x = x(u)$, $u \in U$, where u_1 and u_2 are σ_1-isothermal parameters. Let also, $\widetilde{S}$ be a reparameterization of S defined by a σ_1-diffeomrphism $u = u(\widetilde{u})$, $\widetilde{u} \in \mathbb{T}_{(1)} \times \mathbb{T}_{(2)}$. Then $\widetilde{u}_1$ and $\widetilde{u}_2$ are σ_1-isothermal parameters if and only if the map $u(\widetilde{u})$ is either conformal or anticonformal.*

Proof. Since u_1 and u_2 are σ_1-isothermal parameters, we have

$$F_{1ij} = \lambda^2 \delta_{ij}, \quad i, j \in \{1, 2\}.$$

By the equation (2.3)(see Section 2.1), we have

$$\widetilde{G} = P^T G P,$$

whereupon

$$\widetilde{G} = \lambda^2 P^T P.$$

Thus, $\widetilde{u}_1$ and $\widetilde{u}_2$ are σ_1-isothermal if and only if

$$\widetilde{F}_{1ij} = \widetilde{\lambda}^2 \delta_{ij}, \quad i, j \in \{1, 2\},$$

or if and only if $\dfrac{\lambda}{\widetilde{\lambda}} P$ is an orthogonal matrix, or if and only if $u(\widetilde{u})$ is conformal or anticonformal. This completes the proof.

2.5 The σ_1-Bernstein Theorem

In this section, we will prove some results related to so-called σ_1-Bernstein theorem. We will start with the following useful results.

Theorem 2.11. *Let $f \in \mathscr{C}^1(U)$ and f be a real-valued function. Then, necessary and sufficient condition that the surface*

$$S : x_3 = f(x_1, x_2), \quad (x_1, x_2) \in U,$$

to be on a plane is that there exists a σ_1-nonsingular linear transformation $(u_1, u_2) \to (x_1, x_2)$, $(u_1, u_2) \in \mathbb{T}_{(1)} \times \mathbb{T}_{(2)}$, such that u_1 and u_2 are σ_1-isothermal parameters on S.

Proof. 1. Let such parameters u_1 and u_2 exist. Let also,

$$\zeta = u_1 + i u_2,$$

$$\phi_k(\zeta) = \frac{\partial x_k}{\Delta u_1} - i \frac{\partial x_k}{\Delta u_2}^{\sigma_1}, \quad k \in \{1, 2, 3\}..$$

Since x_1 and x_2 are linear with respect to u_1 and u_2, then we get that ϕ_1 and ϕ_2 are linear functions of u_1 and u_2. Because u_1 and u_2 are in terms of σ_1-isothermal parameters, we have that

$$F_{111} = F_{122},$$

$$F_{112} = 0,$$

whereupon

$$\sum_{k=1}^{3} (\phi_k(\zeta))^2 = F_{111} - F_{122} - 2iF_{112}$$

$$= 0.$$

From here, we conclude that ϕ_3 is also linear. Thus, ϕ_3 has a constant gradient with respect to u_1 and u_2 and hence, also a constant σ_1-gradient with respect to x_1 and x_2. Therefore

$$f(x_1,x_2) = Ax_1 + Bx_2 + C, \quad (x_1,x_2) \in U. \tag{2.26}$$

2. Let f has the form (2.26). Set

$$\lambda^2 = \frac{1}{1+A^2+B^2}$$

and

$$x_1 = \lambda Au_1 + Bu_2,$$

$$x_2 = \lambda Bu_1 - Au_2, \quad (x_1,x_2) \in U,$$

and

$$g(x_1,x_2) = (x_1,x_2,f(x_1,x_2)), \quad (x_1,x_2) \in U.$$

Then

$$\frac{\partial x_1}{\Delta u_1} = \lambda A,$$

$$\frac{\partial x_1}{\Delta u_2} = B,$$

$$\frac{\partial x_2}{\Delta u_1} = \lambda B,$$

$$\frac{\partial x_2}{\Delta u_2} = -A$$

and

$$
\begin{aligned}
\frac{\partial f}{\Delta u_1} &= \frac{\partial f}{\Delta x_1}\frac{\partial x_1}{\Delta u_1} + \frac{\partial f}{\Delta x_2}^{\sigma_1}\frac{\partial x_2}{\Delta u_1} \\
&= \lambda A^2 + \lambda B^2 \\
&= \lambda (A^2 + B^2), \\
\frac{\partial f}{\Delta u_2} &= \frac{\partial f}{\Delta x_1}\frac{\partial x_1}{\Delta u_2} + \frac{\partial f}{\Delta x_2}^{\sigma_1}\frac{\partial x_2}{\Delta u_2} \\
&= AB - AB \\
&= 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\partial g}{\Delta u_1} &= \left(\frac{\partial x_1}{\Delta u_1}, \frac{\partial x_2}{\Delta u_1}, \frac{\partial f}{\Delta u_1}\right) \\
&= (\lambda A, \lambda B, \lambda (A^2 + B^2)), \\
\frac{\partial g}{\Delta u_2}^{\sigma_1} &= \left(\frac{\partial x_1}{\Delta u_2}^{\sigma_1}, \frac{\partial x_2}{\Delta u_2}^{\sigma_1}, \frac{\partial f}{\Delta u_2}^{\sigma_1}\right) \\
&= (B, -A, 0).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
F_{111} &= \frac{\partial g}{\Delta u_1}\frac{\partial g}{\Delta u_1} \\
&= (\lambda A)^2 + (\lambda B)^2 + \lambda^2 (A^2 + B^2)^2 \\
&= \lambda^2 (A^2 + B^2) + \lambda^2 (A^2 + B^2)^2 \\
&= \lambda^2 (A^2 + B^2)(1 + A^2 + B^2) \\
&= \frac{1}{1 + A^2 + B^2}(A^2 + B^2)(1 + A^2 + B^2) \\
&= A^2 + B^2,
\end{aligned}
$$

$$F_{122} = \frac{\partial g}{\Delta u_2}^{\sigma_1} \frac{\partial g}{\Delta u_2}^{\sigma_1}$$
$$= A^2 + B^2$$
$$= F_{111}$$

and

$$F_{112} = \frac{\partial g}{\Delta u_1} \frac{\partial g}{\Delta u_2}^{\sigma_1}$$
$$= (\lambda A)B + \lambda B(-A)$$
$$= 0.$$

Therefore u_1 and u_2 are in terms of σ_1-isothermal parameters on S. This completes the proof.

Theorem 2.12. *Let f be a solution to the minimal surface equation* (2.10) *in the whole x_1, x_2-plane. Suppose that the transformation* (2.23) *is a σ_1-diffeomorphism of the x_1, x_2-plane onto the entire ξ_1, ξ_2-plane. Let also, the functions*

$$\phi_k(\zeta) = \frac{\partial x_1}{\Delta \xi_1} - i\frac{\partial x_k}{\Delta \xi_2}^{\sigma_1}, \quad k \in \{1,2,3\},$$

be such that $\phi_1 \neq 0$, $\phi_2 \neq 0$ and $\frac{\phi_2}{\phi_1}$ be a complex constant. Then, there exists a nonsingular linear transformation

$$\begin{aligned} x_1 &= u_1 \\ x_2 &= au_1 + bu_2, \end{aligned} \tag{2.27}$$

such that u_1, u_2 are global σ_1-isothermal parameters for the surface S defined by

$$x_3 = f(x_1, x_2).$$

Proof. By Theorem 2.9, it follows that ξ_1, ξ_2 are in terms of σ_1-isothermal parameters on the surface S defined by

$$x_3 = f_3(x_1, x_2).$$

Since $\frac{\phi_2}{\phi_1}$ is a constant, there are $a, b \in \mathbb{R}$ such that

$$\phi_2 = (a - ib)\phi_1$$

or

$$\begin{aligned}
\left(\frac{\partial x_2}{\Delta \xi_1} - i\frac{\partial x_2}{\Delta \xi_2}^{\sigma_1}\right) &= (a-ib)\left(\frac{\partial x_1}{\Delta \xi_1} - i\frac{\partial x_1}{\Delta \xi_2}^{\sigma_1}\right)\\
&= a\frac{\partial x_1}{\Delta \xi_1} - ai\frac{\partial x_1}{\Delta \xi_2}^{\sigma_1} - ib\frac{\partial x_1}{\Delta \xi_1} - b\frac{\partial x_1}{\Delta \xi_2}^{\sigma_1}\\
&= a\frac{\partial x_1}{\Delta \xi_1} - b\frac{\partial x_1}{\Delta \xi_2}^{\sigma_1} - i\left(a\frac{\partial x_1}{\Delta \xi_2}^{\sigma_1} + b\frac{\partial x_1}{\Delta \xi_1}\right),
\end{aligned}$$

whereupon

$$\begin{aligned}
\frac{\partial x_2}{\Delta \xi_1} &= a\frac{\partial x_1}{\Delta \xi_1} - b\frac{\partial x_1}{\Delta \xi_2}^{\sigma_1}\\
\frac{\partial x_2}{\Delta \xi_2}^{\sigma_1} &= a\frac{\partial x_1}{\Delta \xi_2}^{\sigma_1} + b\frac{\partial x_1}{\Delta \xi_1}.
\end{aligned} \tag{2.28}$$

Now, we consider the transformation (2.27). Then

$$\begin{aligned}
\frac{\partial x_1}{\Delta \xi_1} &= \frac{\partial x_1}{\Delta u_1}\frac{\partial u_1}{\Delta \xi_1}\\
&= \frac{\partial u_1}{\Delta \xi_1},\\
\frac{\partial x_1}{\Delta \xi_2} &= \frac{\partial x_1}{\Delta u_1}\frac{\partial u_1}{\Delta \xi_2}\\
&= \frac{\partial u_1}{\Delta \xi_2},\\
\frac{\partial x_2}{\Delta \xi_1} &= \frac{\partial x_2}{\Delta u_1}\frac{\partial u_1}{\Delta \xi_1} + \frac{\partial x_2}{\Delta u_2}^{\sigma_1}\frac{\partial u_2}{\Delta \xi_1}\\
&= a\frac{\partial u_1}{\Delta \xi_1} + b\frac{\partial u_2}{\Delta \xi_1},\\
\frac{\partial x_2}{\Delta \xi_2} &= \frac{\partial x_2}{\Delta u_1}\frac{\partial u_1}{\Delta \xi_2} + \frac{\partial x_1}{\Delta u_2}^{\sigma_1}\frac{\partial u_2}{\Delta \xi_2}\\
&= a\frac{\partial u_1}{\Delta \xi_2} + b\frac{\partial u_2}{\Delta \xi_2}.
\end{aligned}$$

Hence and (2.29), we find

$$a\frac{\partial u_1}{\Delta \xi_1} + b\frac{\partial u_2}{\Delta \xi_1} = a\frac{\partial u_1}{\Delta \xi_1} - b\frac{\partial u_1}{\Delta \xi_2}^{\sigma_1}$$

$$a\frac{\partial u_1}{\Delta \xi_2}^{\sigma_1} + b\frac{\partial u_2}{\Delta \xi_2}^{\sigma_1} = a\frac{\partial u_1}{\Delta \xi_2}^{\sigma_1} + b\frac{\partial u_1}{\Delta \xi_1},$$

whereupon

$$\begin{aligned} \frac{\partial u_2}{\Delta \xi_1} &= -\frac{\partial u_1}{\Delta \xi_2}^{\sigma_1} \\ \frac{\partial u_2}{\Delta \xi_2}^{\sigma_1} &= \frac{\partial u_1}{\Delta \xi_1}. \end{aligned} \tag{2.29}$$

From here,

$$\begin{aligned} \frac{\partial u}{\Delta \xi_1} &= \left(\frac{\partial u_1}{\Delta \xi_1}, \frac{\partial u_2}{\Delta \xi_1}\right), \\ \frac{\partial u}{\Delta \xi_2}^{\sigma_1} &= \left(\frac{\partial u_1}{\Delta \xi_2}^{\sigma_1}, \frac{\partial u_2}{\Delta \xi_2}^{\sigma_1}\right) \end{aligned}$$

and

$$\begin{aligned} F_{111} &= \frac{\partial u}{\Delta \xi_1}\frac{\partial u}{\Delta \xi_1} \\ &= \left(\frac{\partial u_1}{\Delta \xi_1}\right)^2 + \left(\frac{\partial u_2}{\Delta \xi_1}\right)^2, \\ F_{122} &= \left(\frac{\partial u_1}{\Delta \xi_2}^{\sigma_1}\right)^2 + \left(\frac{\partial u_2}{\Delta \xi_2}^{\sigma_1}\right)^2 \\ &= \left(-\frac{\partial u_2}{\Delta \xi_1}\right)^2 + \left(\frac{\partial u_1}{\Delta \xi_1}\right)^2 \\ &= F_{111} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial u}{\Delta \xi_1}\frac{\partial u}{\Delta \xi_2}^{\sigma_1} &= \frac{\partial u_1}{\Delta \xi_1}\frac{\partial u_2}{\Delta \xi_2}^{\sigma_1} + \frac{\partial u_2}{\Delta \xi_1}\frac{\partial u_2}{\Delta \xi_2}^{\sigma_1} \\ &= -\frac{\partial u_1}{\Delta \xi_1}\frac{\partial u_2}{\Delta \xi_1} + \frac{\partial u_2}{\Delta \xi_1}\frac{\partial u_1}{\Delta \xi_1} \\ &= 0. \end{aligned}$$

Thus, u_1 and u_2 are global σ_1-isothermal parameters on S. This completes the proof.

Definition 2.28. The system (2.29) is said to be σ_1-Cauchy-Riemann system.

Corollary 2.4 (The σ_1-Bernstein Theorem). *Suppose that all conditions of Theorem 2.12 hold. Then the only solution of the minimal surface equation* (2.10) *in the whole x_1, x_2-plane is the trivial solution, i.e., f is a linear function.*

Proof. By Theorem 2.12, it follows that there exists a nonsingular linear transformation (2.27) such that u_1 and u_2 are σ_1-isothermal parameters. Hence and Theorem 2.11, we conclude that the surface lie on a plane. This completes the proof.

Chapter 3
Global Theory

Suppose that $\mathbb{T}$, $\mathbb{T}_1$, $\mathbb{T}_2$, $\mathbb{T}_3$, $\mathbb{T}_{(j)}$, $j \in \{1,\ldots,n\}$, are time scales with forward jump operators and delta differentiation operators σ, σ_1, σ_2, σ_3, $\sigma_{(j)}$, $j \in \{1,\ldots,n\}$, and Δ, Δ_1, Δ_2, Δ_3, $\Delta_{(j)}$, $j \in \{1,\ldots,n\}$, respectively. Let $I \subseteq \mathbb{T}$, $U, U_1, W_1 \subseteq \mathbb{T}_1 \times \mathbb{T}_2$, $\widetilde{U}$, $W_2 \subset \mathbb{T}_{(1)} \times \ldots \times \mathbb{T}_{(n)}$ and $V \subseteq \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{T}_3$.

3.1 Parametric Surfaces

In the previous chapter, we were able to obtain global results because some special circumstances we had a global parametrization in terms of two of the coordinates. In the general case, we have a surface covered by neighborhoods in each of which a parametrization is given.

In order to study the whole surface we have to first give some important definitions.

Definition 3.1. A σ_1-n-dimensional manifold is a Hausdorff space each point of which has a neighborhood σ_1-homeomorphic to a domain in $\mathbb{T}_{(1)} \times \ldots \times \mathbb{T}_{(n)}$.

Definition 3.2. A σ_1-atlas A for a σ_1-n-dimensional manifold M is a collection of triples $(R_\alpha, O_\alpha, F_\alpha)$, where R_α is a domain in $\mathbb{T}_{(1)} \times \ldots \times \mathbb{T}_{(n)}$, O_α is an open set on M and F_α is a σ_1-homomorphism of $\mathbb{R}_\alpha$ onto O_α, and the union of all the O_α equals M. Each triple is called a σ_1-map.

Definition 3.3. A σ_1-n-manifold is said to be orientable if it possesses a σ_1-atlas for which each transformation $F_\alpha^{-1} \circ F_\beta$ preserves orientation wherever it is defined. An orientation of M is the choice of such a σ_1-atlas.

Definition 3.4. A $\mathscr{C}^r$-structure on M is a σ_1-atlas for which $F_\alpha^{-1} \circ F_\beta \in \mathscr{C}^r$.

Definition 3.5. A conformal structure on a σ_1-n-manifold M is a σ_1-atlas for which $F_\alpha^{-1} \circ F_\beta$ is a conformal map wherever it is defined.

Definition 3.6. Let M be a σ_1-n-manifold with $\mathscr{C}^r$-structure A, and $\widetilde{M}$ be a σ_1-m-manifold with a $\mathscr{C}^r$-structure $\widetilde{A}$. A map $f : M \to \widetilde{M}$ is said to be a $\mathscr{C}^p$-map, denoted by $f \in \mathscr{C}^p$, for $p \leq r$, if each map $\widetilde{F}_\beta \circ f \circ F_\alpha \in \mathscr{C}^p$, wherever it is defined.

Note that, in particular $T_{(1)} \times \mathbb{T}_{(n)}$ has a canonical $\mathscr{C}^r$-structure for all r, defined by letting A consist of the single triple

$$R_\alpha = O_\alpha = \mathbb{T}_{(1)} \times \ldots \times \mathbb{T}_{(n)}$$

and F_α the identity map.

Definition 3.7. A $\mathscr{C}^r$-surface S in $\mathbb{T}_{(1)} \times \ldots \times \mathbb{T}_{(n)}$ is a σ_1-2-manifold M with a $\mathscr{C}^r$ structure, together with a $\mathscr{C}^r$-map $x(p)$ of M into $\mathbb{T}_{(1)} \times \ldots \times \mathbb{T}_{(n)}$.

Suppose that S is a $\mathscr{C}^r$-surface in $\mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$, A the $\mathscr{C}^r$-structure on the associated σ_1-2-manifold M, R_α a domain in the u-plane, and R_β a domain in the $\widetilde{u}$-plane. Then the composition $F_\alpha \circ x(p)$ is a map $x(u) : R_\alpha \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$ which defines a local surface in the sense of Section 2.1. The corresponding map $x(\widetilde{u}) : R_\beta \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$ defines a local surface obtained from $x(u)$ by the change of parameters $u(\widetilde{u}) = F_\alpha^{-1} \circ F_\beta$. Thus, all local properties of surfaces which are independent of parameters are well defined on a global surface S given by the above definition. By a point of S we will mean the pair $(p_0, x(p_0))$, where $p_0 \in M$, and we may speak of S being σ_1-regular of a point, or of the tangent plane and the main curvature vector of S at a point, and so on. The global properties of S will be defined to be those of M. Thus, S will be called orientable if M is orientable, and an orientation of S is an orientation of M. Similarly, for S compact, connected, simply connected and so on.

Definition 3.8. A σ_1-regular $\mathscr{C}^2$-surface S in $\mathbb{T}_{(1)} \times \ldots \times \mathbb{T}_{(n)}$ is a minimal surface if its mean curvature vector vanishes at each point.

Theorem 3.1. *Let S be a σ_1-regular minimal surface in $\mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$ defined by a map $x(p) : M \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$. Then S induces a conformal structure on M.*

Proof. Without loss of generality, suppose that S is orientable. Let A be an orientable atlas of M. Let also, $\widetilde{A}$ be the collection of all triples $(\widetilde{R}_\alpha, \widetilde{O}_\alpha, \widetilde{F}_\alpha)$ such that $\widetilde{R}_\alpha$ is a plane domain, $\widetilde{O}_\alpha$ is an open set in M, $\widetilde{F}_\alpha$ is a σ_1-homomorphism of $\widetilde{R}_\alpha$ onto $\widetilde{O}_\alpha$, $F_\beta^{-1} \circ \widetilde{F}_\alpha$ preserves orientation wherever defined, and

$$x \circ \widetilde{F}_\alpha : \widetilde{R}_\alpha \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$$

a local surface in σ_1-isothermal parameters. By Theorem 2.9, it follows that the union of the $\widetilde{O}_\alpha$ equals M, so that $\widetilde{A}$ is a σ_1-atlas for M. Hence, we get that each $\widetilde{F}_\alpha^{-1} \circ \widetilde{F}_\beta$ is conformal wherever is defined, so that $\widetilde{A}$ defines a conformal structure on M. This completes the proof.

Definition 3.9. We say that a σ_1-harmonic function on a σ_1-n-manifold M has the min-max property if it achieves its maximum and minimum on the boundary of M.

If a σ_1-harmonic function with the min-max property on a σ_1-n-manifold attains its maximum or minimum at an entire point of M, then it is a constant on M.

Definition 3.10. A generalized minimal surface S in $\mathbb{T}_{(1)} \times \ldots \times \mathbb{T}_{(n)}$ is a non-constant map $x(p) : M \to \mathbb{T}_{(1)} \times \ldots \times \mathbb{T}_{(n)}$, where M is a σ_1-2-manifold with a conformal structure defined by a σ_1-atlas $A = \{(R_\alpha, O_\alpha, F_\alpha)\}$ such that each coordinate function $x_k(p)$, $k \in \{1, \ldots, n\}$, is a σ_1-harmonic function on M and

$$\sum_{k=1}^{n} (\phi_k(\zeta))^2 = 0,$$

where

$$h_k(\zeta) = x_k(F_\alpha(\zeta)),$$

$$\phi_k(\zeta) = \frac{\partial h_k}{\Delta \xi_1} - i \frac{\partial h_k}{\Delta \xi_2},$$

$$\zeta = \xi_1 + i\xi_2.$$

Definition 3.11. If the coordinate functions of a generalized minimal surface have the min-max property, we say that the generalized minimal surface has the min-max property.

Let $n = 3$. Then, if S is a σ_1-regular minimal surface, using the conformal structure constructed in Theorem 3.1, we conclude that S is a generalized minimal surface. Thus, the theory of generalized minimal surfaces includes that of σ_1-regular minimal surfaces. On the other hand, if S is a generalized minimal surface, using that the map $x(p)$ is non-constant, we conlcude that at least one of the coordinate functions $x_k(p)$ is non-constant. This implies that the corresponding functions $\phi_k(\zeta)$ can have at most isolated zeros. Therefore the equation

$$\sum_{k=1}^{3} |(\phi_k(\zeta))^2| = 0 \tag{3.1}$$

can hold at most at isolated points. Again applying Theorem 2.8, if we delete these isolated points from S, the reminder of the surface S is a σ_1-regular minimal surface.

Definition 3.12. Let $n = 3$. The points where the equation (3.1) holds are called branch points of the surface.

Theorem 3.2. *Let $n = 3$. A generalized minimal surface that has the min-max property can not be compact.*

Proof. Let S be a generalized minimal surface whose coordinate functions have the min-max property. Let also, S be defined by the map $x(p) : M \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$.

Then each coordinate function $x_k(p)$, $k \in \{1,2,3\}$, is σ_1-harmonic on M with the min-max property. If we assume that M is compact, then $x_k(p)$, $k \in \{1,2,3\}$, would attain its maximum and minimum at entire points of M. Hence, $x_k(p)$, $k \in \{1,2,3\}$, would be a constant. This is a contradiction. This completes the proof.

Definition 3.13. Let M be a σ_1-n-manifold with $\mathscr{C}^r$-structure defined by a σ_1-atlas $S = \{(R_\alpha, O_\alpha, F_\alpha)\}$. A Riemannian structure on M or $\mathscr{C}^q$-Riemannian metric is a collection of matrices G_α, where the elements of G_α are $\mathscr{C}^q$-functions on O_α, $0 \leq q \leq r-1$, and at each point the matrix G_α is positive definite, while for any α, β such that

$$u(\widetilde{u}) = F_\alpha^{-1} \circ F_\beta$$

is defined, the relation

$$G_\beta U^T = G_\alpha U \tag{3.2}$$

must hold, where U is the Jacobian matrix of the transformation $F_\alpha^{-1} \circ F_\beta$.

Definition 3.14. A differentiable curve on a σ_1-n-manifold M is a differentiable map $p(t)$ of an interval $[a,b]$ of $\mathbb{T}$ into M.

Now, we suppose that a Riemannian structure on M is given by the collection of matrices $G_\alpha = (g_{ij})$ and $p(t)$ is a differentiable map on $[a,b] \subset \mathbb{T}$. For each $t_0 \in [a,b]$, we choose an O_α and we set

$$h(t) = \left(\sum_{i,j=1}^{n} g_{ij}(p(t)) u_i^\Delta(t) u_j^\Delta(t) \right)^{\frac{1}{2}}$$

for t sufficiently close to t_0, u_1 and u_2 are coordinates in R_α. Note that by the equation (3.2), it follows that $h(t)$ is independent of the choice of O_α.

Definition 3.15. The length of the curve $p(t)$, $t \in [a,b]$, is defined to be the number

$$\int_a^b h(t) \Delta t.$$

Definition 3.16. Suppose that $0 \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$. A divergent path on M is a continuous map $p(t)$, $t \geq 0$, of the nonnegative elements of $\mathbb{T}$ into M such that for every compact subset Q of M there exists t_0 such that $p(t) \notin Q$ for $t > t_0$. If a divergent path is differentiable, we define its length to be the number

$$\int_0^\infty h(t) \Delta t. \tag{3.3}$$

Definition 3.17. Suppose that $0 \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$. A σ_1-n-manifold M is said to be complete with respect to a given Riemannian metric if the integral (3.3) diverges for any differentiable divergent path on M.

Let now, a $\mathscr{C}^r$-surface S in $\mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$ be defined by a map $x(p) : M \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$, which is σ_1-completely delta differentiable. Then this map induces a Riemannian structure on M, where for each α we set

$$x(u) = x(F_\alpha(u))$$

and we define G_α to be the matrix whose elements are

$$g_{ij} = \frac{\partial x}{\Delta x_i} \cdot \frac{\partial x}{\Delta x_j} \tag{3.4}$$

and the equation (3.2) is a consequence of the equation

$$\widetilde{G} = P^T G P$$

given in Section 2.1, and the matrix G_α will be positive definite at each point where S is σ_1-regular. Thus, to each σ_1-regular surface S in $\mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$ corresponds a Riemannian σ_1-2-manifold M.

Definition 3.18. Let $n = 3$. We say that S is complete if M with is complete with respect to the Riemannian metric defined by (3.4).

3.2 Minimal Surfaces with Boundary

In this section, we will deduct a fundamental property of the minimal surfaces in the case $n = 3$. Suppose that M is a σ_1-3-manifold.

Definition 3.19. A sequnce $\{p_k\}_{k=1}^{\infty}$ of points p_k in M is said to be divergent if it has no points of accumulation on M.

Definition 3.20. If S is a minimal surface defined by a map $x(p) : M \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$, the boundary values of S are the set of points of the form

$$\lim_{k \to \infty} x(p_k)$$

for all divergent sequences $\{p_k\}_{k=1}^{\infty}$ on M.

Remark 3.1. If M is a bounded domain in the plane, then a sequence $\{p_k\}_{k=1}^{\infty}$ in M is divergent if and only if it tends to the boundary of M.

Remark 3.2. If $x(p)$ extends to a continuous map of the closure $\overline{M}$, then the boundary values of S are the image of the boundary of M.

Definition 3.21. A σ_1-3-manifold M is said to have the min-max property if any surface on M has the min-max property.

Theorem 3.3. *Let M has the min-max property. Then any minimal surface lies in the convex hull of its boundary values.*

Proof. Let S be a minimal surface on M. Since M has the min-max property, we have that S has the min-max property. Suppose that S is defined by the map $x(p) : M \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$. In addition, assume that the boundary values of S lie in a half space

$$L(x) = \sum_{k=1}^{3} a_k x_k - b \leq 0.$$

Coinsider the function

$$h(p) = \sum_{k=1}^{3} a_k x_k(p) - b.$$

Since $x_k(p)$, $k \in \{1,2,3\}$, are σ_1-harmonic, we get that $h(p)$ is σ_1-harmonic on M. Let

$$\sup h(p) = m.$$

Then, we choose a sequence $\{p_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} x(p_k) = m.$$

If $\{p_k\}_{k=1}^{\infty}$ has a point of accumulation, then $h(p)$ would assume its maximum at this point and hence, it has to be a constant. If we choose an arbitrary divergent sequence $\{q_k\}_{k=1}^{\infty}$, we will have

$$h(q_k) = m, \quad k \in \mathbb{N},$$

and then

$$\lim_{k\to\infty} h(q_k) \leq 0$$

and from here, $m \leq 0$. On the other hand, if $\{p_k\}_{k=1}^{\infty}$ is divergent, then we have the following

$$m = \lim_{k\to\infty} h(p_k)$$

$$\leq 0.$$

Thus,

$$L(x(p)) \leq 0 \quad \text{on} \quad M$$

and S lies in the half-space $L(x) \leq 0$. The convex hull of the boundary values is the intersection of all the half-spaces which contain them, and S lies in this intersection. This completes the proof.

3.3 Parametric Surfaces in $\mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$. The Gauss Map

Here, in this section, we will discuss several results for the case $n = 3$. Without loss of generality, supoose that $0, 1 \in \mathbb{T}_j, \mathbb{T}_{(j)}$, $j \in \{1, 2, 3\}$, contain negative elements and elements in the interval $(-1, 1)$. Let M be a σ_1-2-manifold.

Definition 3.22. The sets

$$\mathbb{C} = \mathbb{T}_j + i\mathbb{T}_l, \quad j, l \in \{1, 2, 3, (1), (2), (3)\},$$

will be called time scale complex plane, shortly complex plane. The extended time scale complex plane or shortly the extended complex plane is the complex plain plus a point at infinity.

The Riemann sphere can be visualised as the unit sphere

$$x_1^2 + x_2^2 + x_3^2 = 1 \quad \text{in} \quad \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{T}_3$$

with confomal structure defined by a pair of maps

$$F_1 = \left(\frac{2u_1}{|w|^2 + 1}, \frac{2u_2}{|w|^2 + 1}, \frac{|w|^2 - 1}{|w|^2 + 1} \right), \quad w = u_1 + iu_2,$$

$u_1 \in \mathbb{T}_1$, $u_2 \in \mathbb{T}_2$, and

$$F_2 = \left(\frac{2\widetilde{u}_1}{|\widetilde{w}|^2 + 1}, \frac{2\widetilde{u}_2}{|\widetilde{w}|^2 + 1}, \frac{|\widetilde{w}|^2 - 1}{|\widetilde{w}|^2 + 1} \right), \quad \widetilde{w} = \widetilde{u} + i\widetilde{u}_2,$$

$\widetilde{u}_1 \in \mathbb{T}_{(1)}$, $\widetilde{u}_2 \in \mathbb{T}_{(2)}$.

Definition 3.23. The map F_1 is called stereographic projection from the point $(0, 0, 1)$, the image being the whole sphere minus this point.

Let

$$F_1 = (x_1, x_2, x_3).$$

We will find F^{-1}. We have

$$\frac{2u_1}{|w|^2 + 1} = x_1$$

$$\frac{2u_2}{|w|^2 + 1} = x_2$$

$$\frac{|w|^2 - 1}{|w|^2 + 1} = x_3.$$

Then

$$\frac{u_1}{u_2} = \frac{x_1}{x_2}$$

or

$$u_1 = \frac{x_1}{x_2} u_2,$$

and

$$\begin{aligned} w &= u_1 + i u_2 \\ &= \frac{x_1}{x_2} u_2 + i u_2 \\ &= \left(\frac{x_1}{x_2} + i \right) u_2, \end{aligned}$$

and

$$\begin{aligned} |w|^2 &= \left(\frac{x_1^2}{x_2^2} + 1 \right) u_2^2 \\ &= \frac{x_1^2 + x_2^2}{x_2^2} u_2^2 \\ &= \frac{1 - x_3^2}{x_2^2} u_2^2. \end{aligned}$$

Hence,

$$\begin{aligned} x_3 &= \frac{|w|^2 - 1}{|w|^2 + 1} \\ &= \frac{\frac{1-x_3^2}{x_2^2} u_2^2 - 1}{\frac{1-x_3^2}{x_2^2} u_2^2 + 1} \\ &= \frac{(1 - x_3^2) u_2^2 - x_2^2}{(1 - x_3^2) u_2^2 + x_2^2} \end{aligned}$$

and

$$x_3 (1 - x_3^2) u_2^2 + x_3 x_2^2 = (1 - x_3^2) u_2^2 - x_2^2,$$

or

$$(x_3 - 1)(1 - x_3^2) u_2^2 = -x_2^2 (1 + x_3),$$

or

$$(1 - x_3)^2 u_2^2 = x_2^2.$$

Take

$$u_2 = \frac{x_2}{1 - x_3}.$$

Then

$$u_1 = \frac{x_1}{x_2} u_2 = \frac{x_1}{x_2} \cdot \frac{x_2}{1-x_3} = \frac{x_1}{1-x_3}$$

and the map F^{-1} is given by

$$F^{-1} : w = \frac{x_1 + i x_2}{1 - x_3},$$

and $F^{-1} \circ F_2$ is

$$w = \frac{1}{\widetilde{w}},$$

a conformal map of $0 < |\widetilde{w}| < \infty$ onto $0 < |w| < \infty$.

Definition 3.24. A meromorphic function on M is a complex analytic map of M into the Riemann sphere.

Theorem 3.4. *Let U be a domain in the complex ζ-plane, $g(\zeta)$ be an arbitrary merophormic function in U and $f(\zeta)$ be an analytic function in U having the property that at each point where $g(\zeta)$ has a pole of order m, $f(\zeta)$ has a zero of order at least $2m$. Then the functions*

$$\phi_1 = \frac{1}{2} f(1-g^2), \quad \phi_2 = \frac{i}{2} f(1+g^2), \quad \phi_3 = fg \tag{3.5}$$

satisfy

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 1. \tag{3.6}$$

Conversely, every triple of analytic functions in U satisfying (3.6) *may be represented in the form* (3.5)*, except for*

$$\phi_1 = i\phi_2, \quad \phi_3 = 0.$$

Proof. 1. Consider the functions given by (3.5). Then

$$\phi_1^2+\phi_2^2+\phi_3^2=\left(\frac{1}{2}f(1-g^2)\right)^2+\left(\frac{i}{2}f(1+g^2)\right)^2+f^2g^2$$

$$=\frac{1}{4}f^2(1-2g^2+g^4)-\frac{1}{4}f^2(1+2g^2+g^4)+f^2g^2$$

$$=\frac{1}{4}f^2-\frac{1}{2}f^2g^2+\frac{1}{4}f^2g^4-\frac{1}{4}f^2-\frac{1}{2}f^2g^2-\frac{1}{4}f^2g^4+f^2g^2$$

$$=-\frac{1}{2}f^2g^2-\frac{1}{2}f^2g^2+f^2g^2$$

$$=-f^2g^2+f^2g^2$$

$$=0.$$

2. Conversely, for a given solution of (3.6), we set

$$\begin{aligned} f &= \phi_1-i\phi_2,\\ g &= \frac{\phi_3}{\phi_1-i\phi_2}. \end{aligned} \tag{3.7}$$

Note that the equation (3.6) can be written in the form

$$(\phi_1-i\phi_2)(\phi_1+i\phi_2)=-\phi_3^2.$$

Then

$$\begin{aligned} \phi_1+i\phi_2 &= -\frac{\phi_3^2}{\phi_1-i\phi_2}\\ &= -(\phi_1-i\phi_2)\frac{\phi_3^2}{(\phi_1-i\phi_2)^2}\\ &= -fg^2. \end{aligned}$$

Hence and (3.7), we obtain

$$\begin{aligned} \phi_3 &= g(\phi_1-i\phi_2)\\ &= fg \end{aligned}$$

and

$$\phi_1-i\phi_2=f$$

$$\phi_1 + i\phi_2 = -fg^2,$$

whereupon

$$2\phi_1 = f(1-g^2),$$

or

$$\phi_1 = \frac{1}{2} f(1-g^2),$$

and

$$2i\phi_2 = -f(1+g^2),$$

or

$$\phi_2 = -\frac{1}{2i} f(1+g^2)$$

$$= \frac{i}{2} f(1+g^2).$$

This completes the proof.

Theorem 3.5. *Any simpli-connected minimal surface in $\mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$ in the form $x(\zeta): U \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$, $x_k = x_{k1} - i x_{k2}^{\sigma_1}$, where U is either the disk or the plane and the coordinates x_k being σ_1-harmonic, can be represented in the form*

$$x_k(\zeta) = Re\left(\int_0^{\zeta} \phi_k(z)\Delta z\right) + c_k, \quad k \in \{1,2,3\}, \tag{3.8}$$

where ϕ_k, $k \in \{1,2,3\}$, are defined by (3.5)*, the functions f and g having the properties stated in Theorem 3.4, and the integral being taken along an arbitrary path from the origin to the point ζ. The surface will be σ_1-regular if and only if f satisfies the further property that it vanishes only at the poles of g, and the order of its zero at such a point is exactly twice the order of the pole of g.*

Proof. We set

$$\phi_k = \frac{\partial x_k}{\Delta \xi_1} - i \frac{\partial x_k}{\Delta \xi_2}^{\sigma_1},$$

$$\zeta = \xi_1 + i\xi_2.$$

Then (3.8) holds. We have

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = \left(\frac{\partial x_1}{\Delta \xi_1} - i \frac{\partial x_1}{\Delta \xi_2}^{\sigma_1}\right)^2 + \left(\frac{\partial x_2}{\Delta \xi_1} - i \frac{\partial x_2}{\Delta \xi_2}^{\sigma_1}\right)^2 + \left(\frac{\partial x_3}{\Delta \xi_1} - i \frac{\partial x_3}{\Delta \xi_2}^{\sigma_1}\right)^2$$

$$= \left(\frac{\partial x_1}{\Delta \xi_1}\right)^2 - 2i\frac{\partial x_1}{\Delta \xi_1}\frac{\partial x_1}{\Delta \xi_2}^{\sigma_1} - \left(\frac{\partial x_1}{\Delta \xi_2}^{\sigma_1}\right)^2 + \left(\frac{\partial x_2}{\Delta \xi_1}\right)^2 - 2i\frac{\partial x_2}{\Delta \xi_1}\frac{\partial x_2}{\Delta \xi_2}^{\sigma_1} - \left(\frac{\partial x_2}{\Delta \xi_2}^{\sigma_1}\right)^2$$

$$+ \left(\frac{\partial x_3}{\Delta \xi_1}\right)^2 - 2i\frac{\partial x_3}{\Delta \xi_1}\frac{\partial x_3}{\Delta \xi_2}^{\sigma_1} - \left(\frac{\partial x_3}{\Delta \xi_2}^{\sigma_1}\right)^2$$

$$= \left(\left(\frac{\partial x_1}{\Delta \xi_1}\right)^2 + \left(\frac{\partial x_2}{\Delta \xi_1}\right)^2 + \left(\frac{\partial x_3}{\Delta \xi_1}\right)^2\right) - 2i\left(\frac{\partial x_1}{\Delta \xi_1}\frac{\partial x_1}{\Delta \xi_2}^{\sigma_1} + \frac{\partial x_2}{\Delta \xi_1}\frac{\partial x_2}{\Delta \xi_2}^{\sigma_1} + \frac{\partial x_3}{\Delta \xi_1}\frac{\partial x_3}{\Delta \xi_2}^{\sigma_1}\right)$$

$$- \left(\left(\frac{\partial x_1}{\Delta \xi_2}^{\sigma_1}\right)^2 + \left(\frac{\partial x_2}{\Delta \xi_2}^{\sigma_1}\right)^2 + \left(\frac{\partial x_3}{\Delta \xi_2}^{\sigma_1}\right)^2\right)$$

$$= 0,$$

i.e., the equation (3.6) holds. Now, applying Theorem 3.4, we get the representation (3.5). The surface will fail to be σ_1-regular if and only if $f = 0$ where g is regular or $fg^2 = 0$ where g has a pole. This completes the proof.

Now, suppose that x and ϕ are as in Theorem 3.5. Then the tangent plane of the surface is generated by the vectors

$$\frac{\partial x}{\Delta \xi_1} \quad \text{and} \quad \frac{\partial x}{\Delta \xi_2}^{\sigma_1},$$

where

$$\frac{\partial x}{\Delta \xi_1} - i\frac{\partial x}{\Delta \xi_2}^{\sigma_1} = (\phi_1, \phi_2, \phi_3).$$

We have that

$$F_{1ij} = \lambda^2 \delta_{ij}, \quad i, j \in \{1, 2, 3\},$$

where

$$\lambda^2 = \left|\frac{\partial x}{\Delta \xi_1}\right|^2$$

$$= \left|\frac{\partial x}{\Delta \xi_2}^{\sigma_1}\right|^2$$

$$= \frac{1}{2}\sum_{k=1}^{3}|\phi_k|^2$$

$$= \frac{1}{2}(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2)$$

$$= \frac{1}{2}\left(\frac{1}{4}|f|^2|1-g^2|^2 + \frac{1}{4}|f|^2|1+g^2|^2 + |f|^2|g|^2\right)$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\frac{1}{4}|f|^2(1-g^2)(1-\overline{g}^2) + \frac{1}{4}|f|^2(1+g^2)(1+\overline{g}^2) + |f|^2|g|^2\right) \\
&= \frac{1}{2}\left(\frac{1}{4}|f|^2(1-\overline{g}^2 - g^2 + |g|^4) + \frac{1}{4}|f|^2(1+g^2+\overline{g}^2+|g|^4) + |f|^2|g|^2\right) \\
&= \frac{1}{2}\Bigg(\frac{1}{4}|f|^2 - \frac{1}{4}|f|^2\overline{g}^2 - \frac{1}{4}|f|^2 g^2 + \frac{1}{4}|f|^2|g|^4 \\
&\quad + \frac{1}{4}|f|^2 + \frac{1}{4}|f|^2 g^2 + \frac{1}{4}|f|^2\overline{g}^2 + \frac{1}{4}|f|^2|g|^4 + |f|^2|g|^2\Bigg) \\
&= \frac{1}{2}\left(\frac{1}{2}|f|^2 + \frac{1}{2}|f|^2|g|^4 + |f|^2|g|^2\right) \\
&= \frac{1}{4}|f|^2\left(1+|g|^4+2|g|^2\right) \\
&= \frac{1}{4}|f|^2(1+|g|^2)^2 \\
&= \left(\frac{|f|(1+|g|^2)}{2}\right)^2,
\end{aligned}
$$

i.e.,

$$\lambda^2 = \left(\frac{|f|(1+|g|^2)}{2}\right)^2.$$

Furthermore,

$$\frac{\partial x}{\Delta \xi_1} = (\mathrm{Re}\phi_1, \mathrm{Re}\phi_2, \mathrm{Re}\phi_3),$$

$$\frac{\partial x}{\Delta \xi_2}^{\sigma_1} = (\mathrm{Im}\phi_1, \mathrm{Im}\phi_2, \mathrm{Im}\phi_3)$$

amd

$$
\begin{aligned}
\frac{\partial x}{\Delta \xi_1} \times \frac{\partial x}{\Delta \xi_2}^{\sigma_1} = \Big(&\mathrm{Re}\phi_2\mathrm{Im}\phi_3 - \mathrm{Re}\phi_3\mathrm{Im}\phi_2, \mathrm{Im}\phi_1\mathrm{Re}\phi_3 - \mathrm{Re}\phi_1\mathrm{Im}\phi_3, \\
&\mathrm{Re}\phi_1\mathrm{Im}\phi_2 - \mathrm{Im}\phi_1\mathrm{Re}\phi_2\Big).
\end{aligned}
$$

Observe that

$$\phi_2\overline{\phi}_3 = (\mathrm{Re}\phi_2 + i\mathrm{Im}\phi_2)(\mathrm{Re}\phi_3 - i\mathrm{Im}\phi_3)$$

$$= \mathrm{Re}\phi_2\mathrm{Re}\phi_3 - i\mathrm{Re}\phi_2\mathrm{Im}\phi_3 + i\mathrm{Im}\phi_2\mathrm{Re}\phi_3 + \mathrm{Im}\phi_2\mathrm{Im}\phi_3$$

$$= \mathrm{Re}\phi_2\mathrm{Re}\phi_3 + \mathrm{Im}\phi_2\mathrm{Im}\phi_3 - i(\mathrm{Im}\phi_2\mathrm{Re}\phi_3 - \mathrm{Re}\phi_2\mathrm{Im}\phi_3)$$

and

$$\phi_3\overline{\phi}_1 = (\mathrm{Re}\phi_3 + i\mathrm{Im}\phi_3)(\mathrm{Re}\phi_1 - i\mathrm{Im}\phi_1)$$

$$= \mathrm{Re}\phi_1\mathrm{Re}\phi_3 - i\mathrm{Re}\phi_3\mathrm{Im}\phi_1 + i\mathrm{Im}\phi_3\mathrm{Re}\phi_1 + \mathrm{Im}\phi_1\mathrm{Im}\phi_3$$

$$= \mathrm{Re}\phi_1\mathrm{Re}\phi_3 + \mathrm{Im}\phi_1\mathrm{Im}\phi_3 + i(\mathrm{Im}\phi_3\mathrm{Re}\phi_1 - \mathrm{Re}\phi_3\mathrm{Im}\phi_1),$$

and

$$\phi_1\overline{\phi}_2 = (\mathrm{Re}\phi_1 + i\mathrm{Im}\phi_1)(\mathrm{Re}\phi_2 - i\mathrm{Im}\phi_2)$$

$$= \mathrm{Re}\phi_1\mathrm{Re}\phi_2 - i\mathrm{Re}\phi_1\mathrm{Im}\phi_2 + i\mathrm{Im}\phi_1\mathrm{Re}\phi_2 + \mathrm{Im}\phi_1\mathrm{Im}\phi_2$$

$$= \mathrm{Re}\phi_1\mathrm{Re}\phi_2 + \mathrm{Im}\phi_1\mathrm{Im}\phi_2 + i(\mathrm{Im}\phi_1\mathrm{Re}\phi_2 - \mathrm{Re}\phi_1\mathrm{Im}\phi_2).$$

Thus,

$$\frac{\partial x}{\Delta \xi_1} \times \frac{\partial x}{\Delta \xi_2}^{\sigma_1} = \mathrm{Im}(\phi_2\overline{\phi}_3, \phi_3\overline{\phi}_1, \phi_1\overline{\phi}_2).$$

By (3.5), we find

$$\phi_2\overline{\phi}_3 = \frac{i}{2}(\mathrm{Re}f + i\mathrm{Im}f)(1 + (\mathrm{Re}g + i\mathrm{Im}g)^2)$$

$$\overline{(\mathrm{Re}f + i\mathrm{Im}f)(\mathrm{Re}g + i\mathrm{Im}g)}$$

$$= \frac{i}{2}(\mathrm{Re}f + i\mathrm{Im}\phi)(\mathrm{Re}f - i\mathrm{Im}f)$$

$$(\mathrm{Re}g - i\mathrm{Im}g + (\mathrm{Re}g + i\mathrm{Im}g)(\mathrm{Re}g + i\mathrm{Im}g)(\mathrm{Re}g - i\mathrm{Im}g))$$

$$= \frac{i}{2}|f|^2(\mathrm{Re}g - i\mathrm{Im}g + (\mathrm{Re}g + i\mathrm{Im}g)|g|^2)$$

$$= \frac{i}{2}|f|^2(\mathrm{Re}g(1 + |g|^2) - i\mathrm{Im}g(1 - |g|^2))$$

and

$$\mathrm{Im}(\phi_2\overline{\phi}_3) = \frac{|f|^2}{2}\mathrm{Re}g(1+|g|^2).$$

Next,

$$\begin{aligned}
\phi_3\overline{\phi}_1 &= (\mathrm{Re}f+\mathrm{Im}f)(\mathrm{Re}g+i\mathrm{Im}g)\frac{1}{2}\overline{(\mathrm{Re}f+i\mathrm{Im}f)} \\
&\quad (1-\overline{(\mathrm{Re}g+i\mathrm{Im}g)(\mathrm{Re}g+i\mathrm{Im}g)}) \\
&= \frac{1}{2}(\mathrm{Re}f+i\mathrm{Im}f)(\mathrm{Re}f-i\mathrm{Im}f) \\
&\quad (\mathrm{Re}g+i\mathrm{Im}g)(1-(\mathrm{Re}g-i\mathrm{Im}g)(\mathrm{Re}g-i\mathrm{Im}g)) \\
&= \frac{1}{2}|f|^2(\mathrm{Re}g+i\mathrm{Im}g-(\mathrm{Re}g+i\mathrm{Im}g)(\mathrm{Re}g-i\mathrm{Im}g)(\mathrm{Re}g-i\mathrm{Im}g)) \\
&= \frac{1}{2}|f|^2(\mathrm{Re}g+i\mathrm{Im}g-|g|^2(\mathrm{Re}g-i\mathrm{Im}g)) \\
&= \frac{1}{2}|f|^2(\mathrm{Re}g(1-|g|^2)+i\mathrm{Im}g(1+|g|^2)),
\end{aligned}$$

whereupon

$$\mathrm{Im}(\phi_3\overline{\phi}_1) = \frac{1}{2}|f|^2\mathrm{Im}g(1+|g|^2).$$

Moreover,

$$\begin{aligned}
\phi_1\overline{\phi}_2 &= \frac{1}{2}(\mathrm{Re}f+i\mathrm{Im}f)(1-(\mathrm{Re}g+i\mathrm{Im}g)^2) \\
&\quad \left(-\frac{i}{2}\right)\overline{(\mathrm{Re}f+i\mathrm{Im}f)}(1+\overline{(\mathrm{Re}g+i\mathrm{Im}g)}^2) \\
&= -\frac{i}{4}|f|^2\Big(1+(\mathrm{Re}g-i\mathrm{Im}g)^2-(\mathrm{Re}g+i\mathrm{Im}g)^2 \\
&\quad -(\mathrm{Re}g+i\mathrm{Im}g)^2(\mathrm{Re}g-i\mathrm{Im}g)^2\Big) \\
&= -\frac{i}{4}|f|^2\Big(1+(\mathrm{Re}g)^2-2i\mathrm{Re}g\mathrm{Im}g-(\mathrm{Im}g)^2-(\mathrm{Re}g)^2-2i\mathrm{Re}g\mathrm{Im}g \\
&\quad +(\mathrm{Im}g)^2-|g|^4\Big) \\
&= -\frac{i}{4}|f|^2(1-|g|^4-4i\mathrm{Re}g\mathrm{Im}g)
\end{aligned}$$

$$= -\frac{i}{4}|f|^2((1-|g|^2)(1+|g|^2) - 4i\mathrm{Re}g\mathrm{Im}g),$$

i.e.,

$$\mathrm{Im}(\phi_1\overline{\phi}_2) = \frac{1}{4}|f|^2(|g|^2-1)(|g|^2+1).$$

Consequently

$$\begin{aligned}\frac{\partial x}{\Delta\xi_1} \times \frac{\partial x}{\Delta\xi_2}^{\sigma_1} &= \left(\frac{|f|^2}{2}\mathrm{Re}g(1+|g|^2), \frac{|f|^2}{2}\mathrm{Im}g(1+|g|^2),\right.\\ &\qquad \left.\frac{|f|^2}{4}(|g|^2-1)(|g|^2+1)\right)\\ &= \frac{|f|^2}{4}(1+|g|^2)(2\mathrm{Re}g, 2\mathrm{Im}g, |g|^2-1).\end{aligned}$$

Hence,

$$\begin{aligned}\left|\frac{\partial x}{\Delta\xi_1} \times \frac{\partial x}{\Delta\xi_2}^{\sigma_1}\right| &= \frac{|f|^2}{4}(1+|g|^2)\sqrt{4(\mathrm{Re}g)^2 + 4(\mathrm{Im}g)^2 + (|g|^2-1)^2}\\ &= \frac{|f|^2}{4}(1+|g|^2)\sqrt{4|g|^2 + |g|^4 - 2|g|^2 + 1}\\ &= \frac{|f|^2}{4}(1+|g|^2)\sqrt{|g|^4 + 2|g|^2 + 1}\\ &= \frac{|f|^2}{4}(1+|g|^2)(1+|g|^2)\\ &= \frac{|f|^2}{4}(1+|g|^2)^2\\ &= \left(\frac{|f|(1+|g|^2)}{2}\right)^2\end{aligned}$$

and

$$\begin{aligned}N &= \frac{\frac{\partial x}{\Delta\xi_1} \times \frac{\partial x}{\Delta\xi_2}^{\sigma_1}}{\left|\frac{\partial x}{\Delta\xi_1} \times \frac{\partial x}{\Delta\xi_2}^{\sigma_1}\right|}\\ &= \frac{1}{\frac{|f|^2(1+|g|^2)^2}{4}}\frac{|f|^2}{4}(1+|g|^2)(2\mathrm{Re}g, 2\mathrm{Im}g, |g|^2-1)\end{aligned}$$

$$= \left(\frac{2\mathrm{Re} g}{1+|g|^2}, \frac{2\mathrm{Im} g}{1+|g|^2}, \frac{|g|^2-1}{1+|g|^2} \right),$$

i.e.,

$$N = \left(\frac{2\mathrm{Re} g}{1+|g|^2}, \frac{2\mathrm{Im} g}{1+|g|^2}, \frac{|g|^2-1}{1+|g|^2} \right). \tag{3.9}$$

Definition 3.25. For an arbitrary σ_1-regular surface $x(u)$ in $\mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$ one defines the Gauss map to be the map

$$x(u) \to N(u) = \frac{\frac{\partial x}{\Delta u_1} \times \frac{\partial x}{\Delta u_2}^{\sigma_1}}{\left| \frac{\partial x}{\Delta u_1} \times \frac{\partial x}{\Delta u_2}^{\sigma_1} \right|}$$

of the surface into the unitr sphere.

Theorem 3.6. *Let $x(\zeta) : U \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$ defines a σ_1-regular minimal surface. Then the Gauss map defines a complex analytic map of U into the unit sphere considered as the Riemann sphere.*

Proof. Note that the formula (3.9) cpompared with the stereographic projection F_1 shows that the Gauss map $x(\zeta) \to N(\zeta)$ followed by the stereographic projection F_1 from the point $(0,0,1)$ yields the meromorphic function $g(\zeta)$. This completes the proof.

Theorem 3.7. *Let $x(\zeta) : U \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$ define a generalized minimal surface S, where U is the entire ζ-plane. Then either $x(\zeta)$ lies on a plane, or else the normals to S take on all directions with at most two exceptions.*

Proof. The surface S we associate with the function $g(\zeta)$ which fails to be defined only if

$$\phi_1 = i\phi_2,$$

$$\phi_3 = 0.$$

In this case, we have that x_3 is a constant and the surface lies in a plane. Otherwise, applying Theorem 3.6, we have that $g(\zeta)$ is meromorphic in the entire ζ-plane, and by the Picard theorem it either takes on all values with at most two exceptions, or else it is a constant. By (3.9) the same alternative applies to the normal N and in the latter case S lies on a plane. This completes the proof.

Theorem 3.8. *Let $f(z)$ be an analytic function in the unit disk U which has at most a finite number of zeros. Then there exists a divergent path C in U such that*

$$\int_C |f(z)||\Delta z| < \infty.$$

Proof. Suppose that $f(z) \neq 0$, $z \in U$. Define

$$w = F(z)$$

$$= \int_0^z f(\zeta)\Delta\zeta.$$

Thus, $F(z)$ maps $|z| < 1$ onto a Riemann sphere which has no branch points. If we set

$$z = G(w)$$

to be the branch of the inverse function satisfying $G(0) = 0$, then using that $|G(w)| < 1$, we conclude that there is a largest disk $|w| < R < \infty$ in which $G(w)$ is defined. Hence, there exists a point w_0 so that $|w_0| = R$ and $G(w)$ cannot be extended to a neighbourhood of w_0. Let L be the line segment

$$w = tw_0, \quad 0 \leq t < 1,$$

and C be the image of L under $G(w)$. Then C must be a divergent path. Otherwise, there would be a sequence $\{t_n\}_{n=1}^{\infty}$ so that

$$\lim_{n\to\infty} t_n = 1$$

and the corresponding sequence $\{z_n\}_{n=1}^{\infty}$ would converge to a point $z_0 \in U$. Then

$$F(z_0) = w_0$$

and

$$F^{\Delta}(z_0) = f(z_0)$$

$$\neq 0$$

and the function $G(w)$ would be extended to a neighborhood of w_0. Therefore the path C is divergent and

$$\int_C |f(z)||\Delta z| = \int_0^1 |f(z)| \left|\frac{\Delta z}{\Delta t}\right| \Delta t$$

$$= R$$

$$< \infty.$$

Now, suppose that f has a finite number of zeros, say of order v_k of the points z_k. Consider the function

$$f_1(z)=f(z)\prod_k\left(\frac{1-\bar{z}_k z}{z-z_k}\right)^{v_k}$$

never vanishes and by the above arguments, it follows that there exists a divergent path C so that

$$\int_C |f_1(z)||\Delta z|<\infty.$$

Note that

$$\begin{aligned}|f_1(z)| &= \left|f(z)\prod_k\left(\frac{1-\bar{z}_k z}{z-z_k}\right)^{v_k}\right|\\ &= |f(z)|\prod_k\left|\frac{1-\bar{z}_k z}{z-z_k}\right|^{v_k}, \quad |z|<1,\end{aligned}$$

whereupon

$$|f(z)|=|f_1(z)|\prod_k\left|\frac{z-z_k}{z-\bar{z}_k z}\right|^{v_k}, \quad |z|<1. \tag{3.10}$$

Note that

$$\begin{aligned}z-z_k &= \mathrm{Re}z+\mathrm{Im}z-\mathrm{Re}z_k-i\mathrm{Im}z_k\\ &= (\mathrm{Re}z-\mathrm{re}z_k)+i(\mathrm{Im}z-\mathrm{Im}z_k)\end{aligned}$$

and

$$\begin{aligned}1-\bar{z}_k z &= 1-(\mathrm{Re}z_k-i\mathrm{Im}z_k)(\mathrm{Re}z+i\mathrm{Im}z)\\ &= 1-(\mathrm{Re}z\mathrm{Re}z_k+\mathrm{Im}z_k\mathrm{Im}z-i(\mathrm{Im}z_k\mathrm{Re}z-\mathrm{Im}z\mathrm{Re}z_k))\\ &= (1-\mathrm{Re}z\mathrm{Re}z_k-\mathrm{Im}z_k\mathrm{Im}z)+i(\mathrm{Im}z_k\mathrm{Re}z-\mathrm{Im}z\mathrm{Re}z_k).\end{aligned}$$

Then

$$|1-\bar{z}_k z|^2-|z-z_k|^2$$

$$\begin{aligned}&= (1-\mathrm{Re}z\mathrm{Re}z_k-\mathrm{Im}z_k\mathrm{Im}z)^2+(\mathrm{Im}z_k\mathrm{Re}z-\mathrm{Im}z\mathrm{Re}z)^2\\ &\quad -(\mathrm{Re}z-\mathrm{Re}z_k)^2-(\mathrm{Im}z-\mathrm{Im}z_k)^2\\ &= 1+(\mathrm{Re}z)^2(\mathrm{Re}z_k)^2+(\mathrm{Im}z)^2(\mathrm{Im}z_k)^2-2\mathrm{Re}z\mathrm{Re}z_k-2\mathrm{Im}z\mathrm{Im}z_k\end{aligned}$$

$$+2\mathrm{Re}z\mathrm{Im}z\mathrm{Re}z_k\mathrm{Im}z_k+(\mathrm{Im}z_k)^2(\mathrm{Re}z)^2-2\mathrm{Re}z\mathrm{Re}z_k\mathrm{Im}z\mathrm{Im}z_k$$

$$+(\mathrm{Im}z)^2(\mathrm{Re}z_k)^2-(\mathrm{Re}z)^2+2\mathrm{Re}z\mathrm{Re}z_k-(\mathrm{Re}z_k)^2-(\mathrm{Im}z)^2+2\mathrm{Im}z\mathrm{Im}z_k-(\mathrm{Im}z_k)^2$$

$$=1+|z|^2(\mathrm{Re}z_k)^2+|z|^2(\mathrm{Im}z_k)^2-|z|^2-|z_k|^2$$

$$=1+|z|^2|z_k|^2-|z|^2-|z_k|^2$$

$$=(1-|z|^2)+|z_k|^2(|z|^2-1)$$

$$=(1-|z_k|^2)(1-|z|^2)$$

$$\geq 0, \quad |z|<1,$$

i.e.,

$$|z-z_k|\leq|1-\bar{z}_k z|, \quad |z|<1.$$

Hence and (3.10), we get

$$|f(z)|=|f_1(z)|\prod_k \frac{|z-z_k|^{\nu_k}}{|1-\bar{z}_k z|^{\nu_k}}$$

$$\leq|f_1(z)|, \quad |z|<1.$$

Therefore

$$\int_C |f(z)||\Delta z|\leq\int_C |f_1(z)||\Delta z|$$

$$<\infty.$$

This completes the proof.

Definition 3.26. Let M be a σ_1-2-manifold. If there exists a simply-connected σ_1-2-manifold $\widehat{M}$ and a map $\pi:\widehat{M}\to M$ with the property that each point of M has a neighborhood V such that the restriction of π to each component of $\pi^{-1}(V)$ is a σ_1-hpomomorphism onto V, we say that M has an universal covering surface.

Suppose that M is a σ_1-2-manifold that has an universal covering surface. Then the map π is a local σ_1-homomorphism, and it follows that any structure of M: $\mathscr{C}^r$, conformal, Riemannian and etc., induces a corresponding structure on $\widehat{M}$. Now, we suppose that S is a minimal surface defined by a map $x(p):M\to\mathbb{T}_{(1)}\times\mathbb{T}_{(2)}\times\mathbb{T}_{(3)}$. Then we have an associated simply-connected minimal surface $\widehat{S}$ called the universal covering surface of S, defined by the composed map

$$x(\pi(\widehat{p})) : \widehat{M} \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}.$$

It follows that $\widehat{S}$ is σ_1-regular if and only if S is σ_1-regular, and $\widehat{S}$ is complete if and only if S is complete. Thus, many questions concerning minimal surfaces may be settled by considering only simply-connected minimal surfaces.

Theorem 3.9. *Let S be a complete σ_1-regular minimal surface in $\mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$ so that its universal covering surface $\widehat{S}$ may be represented in the form $x(\zeta) : U \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$, where U is the plane or the unit disk. Then either S is a plane or else the normals to S are everywhere dense.*

Proof. Suppose that the normals to S are not everywhere dense. Then there exists an open set on the unit sphere which is not intersected by the image of S under the Gauss map. By a rotation in space, we may assume that the point $(0,0,1)$ is in this open set. Let the normal N be in the form

$$N = (N_1, N_2, N_3).$$

Then there exists $\eta < 1$ such that

$$N_3 \leq \eta.$$

The same is true for the universal covering surface $\widehat{S}$ of S which may be represented in the form $x(\zeta) : U \to \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$, where U is the plane or the unit disk. Note that U can not be the unit disk since, using (3.9), we have

$$N_3 \leq \eta < 1$$

if and only if

$$|g(\zeta)| \leq M < \infty,$$

and $\widehat{S}$ is σ_1-regular, we have that $f(\zeta)$ cannot vanish. The length of any path C would be

$$\int_C \lambda |\Delta\zeta| = \frac{1}{2} \int_C |f|(1+|g|^2)|\Delta\zeta|$$

$$< \frac{1+M^2}{2} \int_C |f||\Delta\zeta|.$$

By Theorem 3.8, it follows that there exists a divergent path C for which this integral converges and then the surface would not be complete. Thus, U is the entire plane, and since the normals omit more than two points, applying Theorem 3.7, we conclude that $\widehat{S}$ would be on a plane. The same is true of S, and since S is complete, we conclude that S must be the whole plane. This completes the proof.

Theorem 3.10. *Let U be an arbitrary set of k points in the unit disk, where $k \leq 4$. Then there exists a complete σ_1-regular minimal surface in $\mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$ whose image under the Gauss map omits precisely the set U.*

Proof. By a rotation, we may assume that the set U contains the point $(0,0,1)$. If this is only point, then by setting

$$f(\zeta) = 1,$$

$$g(\zeta) = \zeta,$$

solves the problem. Otherwise, let the other points of U correspond to the points w_m, $m \in \{1,\dots,k-1\}$, under stereographic projection. Then, we set

$$f(\zeta) = \frac{1}{\prod_{m=1}^{k-1}(\zeta - w_m)},$$

$$g(\zeta) = \zeta.$$

Then, we will use the representation (3.5), (3.8) in the whole ζ-plane minus the points w_m, to obtain a minimal surface whose normals omit precisely the points of U by (3.9). It will be complete because a divergent path must tend either to ∞ or to one of the points w_m and in either case, we obtain

$$\int_C \lambda|\Delta\zeta| = \frac{1}{2}\int_C |f|(1+|g|^2)|\Delta\zeta|$$

$$= \infty.$$

In the case when the integrals (3.8) may not be single-valued, then by passing to the universal covering surface we get a single-valued map defining a surface having the same properties. This completes the proof.

3.4 The Gauss Curvature. The Total Curvature

In this section, we continue the study of minimal surfaces in $\mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$, using the representation (3.5). Then, for the first fundamental form, we have the following

$$F_{1ij} = \left(\frac{|f|(1+|g|^2)}{2}\right)^2 \delta_{ij}, \quad i,j \in \{1,2\},$$

and

$$\lambda^2 = \left(\frac{\partial x}{\Delta\xi_1}\right)^2$$
$$= \left(\frac{\partial x}{\Delta\xi_2}^{\sigma_1}\right)^2,$$

and

$$F_{111}\left(\frac{d\xi_1}{\Delta t}\right)^2 + F_{122}\left(\frac{d\xi_2}{\Delta t}\right)^2 = F_{111}\left(\left(\frac{d\xi_1}{\Delta t}\right)^2 + \left(\frac{d\xi_2}{\Delta t}\right)^2\right)$$
$$= \left(\frac{|f|(1+|g|^2)}{2}\right)^2 \left|\frac{d\zeta}{\Delta t}\right|^2.$$

Now, we will determine the second fundamental form. We have

$$\frac{\partial^2 x_1}{\Delta\xi_1^2} = \frac{\partial}{\Delta\xi_1}\left(\frac{\partial x_1}{\Delta\xi_1}\right)$$
$$= \frac{\partial}{\Delta\xi_1}(\mathrm{Re}\phi_1)$$
$$= \frac{1}{2}\mathrm{Re}\left(\frac{\partial}{\Delta\xi_1}(f(1-g^2))\right)$$
$$= \frac{1}{2}\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}(1-g^2) + f^{\sigma_1}\left(-g\frac{\partial g}{\Delta\xi_1} - g^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\right)$$
$$= \frac{1}{2}\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}(1-g^2) - f^{\sigma_1}\left(g\frac{\partial g}{\Delta\xi_1} + g^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\right)$$
$$= \frac{1}{2}\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}(1-g^2) - f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}(g+g^{\sigma_1})\right)$$
$$= \frac{1}{2}\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}(1-g^2)\right) - \frac{1}{2}\mathrm{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}(g+g^{\sigma_1})\right)$$

and

$$\frac{\partial^2 x_2}{\Delta\xi_1^2} = \frac{\partial}{\Delta\xi_1}\left(\frac{\partial x_1}{\Delta\xi_1}\right)$$
$$= \frac{\partial}{\Delta\xi_1}(\mathrm{Re}\phi_2)$$
$$= \frac{1}{2}\mathrm{Re}\left(i\frac{\partial}{\Delta\xi_1}(f(1+g^2))\right)$$

$$
\begin{aligned}
&= -\frac{1}{2}\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_1}(1+g^2)+f^{\sigma_1}\left(g\frac{\partial g}{\Delta\xi_1}+g^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\right)\\
&= -\frac{1}{2}\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_1}(1+g^2)+f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}(g+g^{\sigma_1})\right)\\
&= -\frac{1}{2}\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_1}(1+g^2)\right)-\frac{1}{2}\mathrm{Im}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}(g+g^{\sigma_1})\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 x_3}{\Delta\xi_1^2} &= \frac{\partial}{\delta\xi_1}\left(\frac{\partial x_1}{\Delta\xi_1}\right)\\
&= \frac{\partial}{\Delta\xi_1}(\mathrm{Re}\phi_3)\\
&= \mathrm{Re}\left(\frac{\partial}{\Delta\xi_1}(fg)\right)\\
&= \mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}g+f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\\
&= \mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}g\right)+\mathrm{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{\partial^2 x}{\Delta\xi_1^2}\cdot N = &\left(\frac{1}{2}\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}(1-g^2)\right)-\frac{1}{2}\mathrm{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}(g+g^{\sigma_1})\right),\right.\\
&-\frac{1}{2}\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_1}(1+g^2)\right)-\frac{1}{2}\mathrm{Im}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}(g+g^{\sigma_1})\right),\\
&\left.\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}g\right)+\mathrm{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\right)\\
&\cdot\left(\frac{2\mathrm{Re}g}{|g|^2+1},\frac{2\mathrm{Im}g}{|g|^2+1},\frac{|g|^2-1}{|g|^2+1}\right)\\
= &\ \mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}(1-g^2)\right)\frac{\mathrm{Re}g}{|g|^2+1}-\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_1}(1+g^2)\right)\frac{\mathrm{Im}g}{|g|^2+1}\\
&+\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}g\right)\frac{|g|^2-1}{|g|^2+1}\\
&-\mathrm{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}(g+g^{\sigma_1})\right)\frac{\mathrm{Re}g}{|g|^2+1}-\mathrm{Im}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}(g+g^{\sigma_1})\right)\frac{\mathrm{Im}g}{|g|^2+1}
\end{aligned}
$$

$$+\mathrm{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\frac{|g|^2-1}{|g|^2+1}.$$

Let

$$\begin{aligned}
I_{11} &= \mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}(1-g^2)\right)\frac{\mathrm{Re}g}{|g|^2+1}-\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_1}(1+g^2)\right)\frac{\mathrm{Im}g}{|g|^2+1}\\
&\quad+\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}g\right)\frac{|g|^2-1}{|g|^2+1},\\
I_{12} &= -\mathrm{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}(g+g^{\sigma_1})\right)\frac{\mathrm{Re}g}{|g|^2+1}-\mathrm{Im}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}(g+g^{\sigma_1})\right)\frac{\mathrm{Im}g}{|g|^2+1}\\
&\quad+\mathrm{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\frac{|g|^2-1}{|g|^2+1}.
\end{aligned}$$

Then

$$\frac{\partial^2 x}{\Delta\xi_1^2}\cdot N = I_{11}+I_{12}.$$

For I_{11}, we have

$$\begin{aligned}
I_{11} &= \mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}(1-g^2)\right)\frac{\mathrm{Re}g}{|g|^2+1}-\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_1}(1+g^2)\right)\frac{\mathrm{Im}g}{|g|^2+1}\\
&\quad+\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}g\right)\frac{|g|^2-1}{|g|^2+1}\\
&= \frac{1}{|g|^2+1}\left(\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}-\frac{\partial f}{\Delta\xi_1}g^2\right)\mathrm{Re}g-\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_1}+\frac{\partial f}{\Delta\xi_1}g^2\right)\mathrm{Im}g\right.\\
&\quad\left.+\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}g\right)(|g|^2-1)\right)\\
&= \frac{1}{|g|^2+1}\left(\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}\right)\mathrm{Re}g-\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}g^2\right)\mathrm{Re}g-\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_1}\right)\mathrm{Im}g-\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_1}g^2\right)\mathrm{Im}g\right.\\
&\quad\left.+\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}g\right)(|g|^2-1)\right)\\
&= \frac{1}{|g|^2+1}\left(\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}g\right)-\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}g^2\right)\mathrm{Re}g-\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_1}g^2\right)\mathrm{Im}g\right.\\
&\quad\left.+\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}g\right)|g|^2-\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}g\right)\right)\\
&= \frac{1}{|g|^2+1}\left(\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}g^2\right)\mathrm{Re}g-\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_1}g^2\right)\mathrm{Im}g+\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_1}g\right)|g|^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|g|^2+1}\left(-\operatorname{Re}\left(\frac{\partial f}{\Delta\xi_1}g\right)(\operatorname{Re}g)^2+\operatorname{Im}\left(\frac{\partial f}{\Delta\xi_1}g\right)\operatorname{Im}g\operatorname{Re}g-\operatorname{Im}\left(\frac{\partial f}{\Delta\xi_1}g\right)\operatorname{Re}g\operatorname{Im}g\right.\\
&\quad\left.-\operatorname{Re}\left(\frac{\partial f}{\Delta\xi_1}g\right)(\operatorname{Im}g)^2+\operatorname{Re}\left(\frac{\partial f}{\Delta\xi_1}g\right)(\operatorname{Re}g)^2+\operatorname{Re}\left(\frac{\partial f}{\Delta\xi_1}g\right)(\operatorname{Im}g)^2\right)\\
&= 0.
\end{aligned}
$$

Now, we consider I_{12}. We have

$$
\begin{aligned}
I_{12} &= -\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}(g+g^{\sigma_1})\right)\frac{\operatorname{Re}g}{|g|^2+1}-\operatorname{Im}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}(g+g^{\sigma_1})\right)\frac{\operatorname{Im}g}{|g|^2+1}\\
&\quad+\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\frac{|g|^2-1}{|g|^2+1}\\
&= \frac{1}{|g|^2+1}\left(-\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}(g+g^{\sigma_1})\right)\operatorname{Re}g-\operatorname{Im}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}(g+g^{\sigma_1})\right)\operatorname{Im}g\right.\\
&\quad\left.+\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)(|g|^2-1)\right)\\
&= \frac{1}{|g|^2+1}\left(-\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\operatorname{Re}(g+g^{\sigma_1})\operatorname{Re}g+\operatorname{Im}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\operatorname{Im}(g+g^{\sigma_1})\operatorname{Re}g\right.\\
&\quad-\operatorname{Im}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\operatorname{Re}(g+g^{\sigma_1})\operatorname{Im}g-\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\operatorname{Im}(g+g^{\sigma_1})\operatorname{Im}g\\
&\quad\left.+\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)(\operatorname{Re}g)^2+\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)(\operatorname{Im}g)^2-\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\right)\\
&= \frac{1}{|g|^2+1}\left(-\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)(\operatorname{Re}g)^2-\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\operatorname{Re}g^{\sigma_1}\operatorname{Re}g\right.\\
&\quad+\operatorname{Im}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\operatorname{Im}g\operatorname{Re}g+\operatorname{Im}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\operatorname{Im}g^{\sigma_1}\operatorname{Re}g\\
&\quad-\operatorname{Im}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\operatorname{Im}g\operatorname{Re}g-\operatorname{Im}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\operatorname{Re}g^{\sigma_1}\operatorname{Im}g\\
&\quad-\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)(\operatorname{Im}g)^2-\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\operatorname{Im}g^{\sigma_1}\operatorname{Im}g\\
&\quad\left.+\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)(\operatorname{Re}g)^2+\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)(\operatorname{Im}g)^2-\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)\right)
\end{aligned}
$$

$$= \frac{1}{|g|^2+1}\left(-\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta \xi_1}\right)(\operatorname{Re} g^{\sigma_1}\operatorname{Re} g + \operatorname{Im} g^{\sigma_1}\operatorname{Im} g)\right.$$

$$\left. +\operatorname{Im}\left(f^{\sigma_1}\frac{\partial g}{\Delta \xi_1}\right)(\operatorname{Im} g^{\sigma_1}\operatorname{Re} g - \operatorname{Re} g^{\sigma_1}\operatorname{Im} g)\right),$$

i.e.,

$$I_{12} = \frac{1}{|g|^2+1}\left(-\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta \xi_1}\right)(\operatorname{Re} g^{\sigma_1}\operatorname{Re} g + \operatorname{Im} g^{\sigma_1}\operatorname{Im} g)\right.$$

$$\left. +\operatorname{Im}\left(f^{\sigma_1}\frac{\partial g}{\Delta \xi_1}\right)(\operatorname{Im} g^{\sigma_1}\operatorname{Re} g - \operatorname{Re} g^{\sigma_1}\operatorname{Im} g)\right).$$

Consequently

$$\frac{\partial^2 x}{\Delta \xi_1^2}\cdot N = I_{11} + I_{12}$$

$$= \frac{1}{|g|^2+1}\left(-\operatorname{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta \xi_1}\right)(\operatorname{Re} g^{\sigma_1}\operatorname{Re} g + \operatorname{Im} g^{\sigma_1}\operatorname{Im} g)\right.$$

$$\left. +\operatorname{Im}\left(f^{\sigma_1}\frac{\partial g}{\Delta \xi_1}\right)(\operatorname{Im} g^{\sigma_1}\operatorname{Re} g - \operatorname{Re} g^{\sigma_1}\operatorname{Im} g)\right).$$

Next,

$$\frac{\partial^2 x_1}{\Delta \xi_2^2} = \frac{\partial}{\Delta \xi_2}\left(\frac{\partial x_1}{\Delta \xi_2}\right)$$

$$= -\frac{\partial}{\Delta \xi_2}(\operatorname{Re}\phi_1)$$

$$= -\frac{1}{2}\operatorname{Re}\left(\frac{\partial}{\Delta \xi_2}(f(1-g^2))\right)$$

$$= -\frac{1}{2}\operatorname{Re}\left(\frac{\partial f}{\Delta \xi_2}(1-g^2) + f^{\sigma_2}\left(-g\frac{\partial g}{\Delta \xi_2} - g^{\sigma_2}\frac{\partial g}{\Delta \xi_2}\right)\right)$$

$$= -\frac{1}{2}\operatorname{Re}\left(\frac{\partial f}{\Delta \xi_2}(1-g^2) - f^{\sigma_2}\left(g\frac{\partial g}{\Delta \xi_2} + g^{\sigma_2}\frac{\partial g}{\Delta \xi_2}\right)\right)$$

$$= -\frac{1}{2}\operatorname{Re}\left(\frac{\partial f}{\Delta \xi_2}(1-g^2) - f^{\sigma_2}\frac{\partial g}{\Delta \xi_2}(g+g^{\sigma_2})\right)$$

$$= -\frac{1}{2}\operatorname{Re}\left(\frac{\partial f}{\Delta \xi_2}(1-g^2)\right) + \frac{1}{2}\operatorname{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta \xi_2}(g+g^{\sigma_2})\right)$$

and

$$\begin{aligned}
\frac{\partial^2 x_2}{\Delta\xi_2^2} &= \frac{\partial}{\Delta\xi_2}\left(\frac{\partial x_1}{\Delta\xi_2}\right) \\
&= -\frac{\partial}{\Delta\xi_2}\left(\mathrm{Re}\phi_2\right) \\
&= -\frac{1}{2}\mathrm{Re}\left(i\frac{\partial}{\Delta\xi_2}\left(f(1+g^2)\right)\right) \\
&= \frac{1}{2}\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_2}(1-g^2)+f^{\sigma_2}\left(g\frac{\partial g}{\Delta\xi_2}+g^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)\right) \\
&= \frac{1}{2}\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_2}(1-g^2)+f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\left(g+g^{\sigma_2}\right)\right) \\
&= \frac{1}{2}\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_2}(1-g^2)\right)+\frac{1}{2}\mathrm{Im}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\left(g+g^{\sigma_2}\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 x_3}{\Delta\xi_2^2} &= \frac{\partial}{\delta\xi_2}\left(\frac{\partial x_1}{\Delta\xi_2}\right) \\
&= -\frac{\partial}{\Delta\xi_2}(\mathrm{Re}\phi_3) \\
&= -\mathrm{Re}\left(\frac{\partial}{\Delta\xi_2}(fg)\right) \\
&= -\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g+f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right) \\
&= -\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g\right)-\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right).
\end{aligned}$$

From here,

$$\begin{aligned}
\frac{\partial^2 x}{\Delta\xi_2^2}\cdot N = &\left(-\frac{1}{2}\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}(1-g^2)\right)+\frac{1}{2}\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\left(g+g^{\sigma_2}\right)\right),\right. \\
&+\frac{1}{2}\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_2}(1-g^2)\right)+\frac{1}{2}\mathrm{Im}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\left(g+g^{\sigma_2}\right)\right), \\
&\left.-\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g\right)-\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)\right) \\
&\cdot\left(\frac{2\mathrm{Re}g}{|g|^2+1},\frac{2\mathrm{Im}g}{|g|^2+1},\frac{|g|^2-1}{|g|^2+1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}(1-g^2)\right)\frac{\mathrm{Re}g}{|g|^2+1} + \mathrm{Im}\left(\frac{\partial f}{\Delta\xi_2}(1+g^2)\right)\frac{\mathrm{Im}g}{|g|^2+1} \\
&\quad +\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g\right)\frac{|g|^2-1}{|g|^2+1} \\
&\quad +\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}(g+g^{\sigma_2})\right)\frac{\mathrm{Re}g}{|g|^2+1} + \mathrm{Im}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}(g+g^{\sigma_2})\right)\frac{\mathrm{Im}g}{|g|^2+1} \\
&\quad -\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)\frac{|g|^2-1}{|g|^2+1}.
\end{aligned}
$$

Let

$$
\begin{aligned}
I_{21} &= -\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}(1-g^2)\right)\frac{\mathrm{Re}g}{|g|^2+1} + \mathrm{Im}\left(\frac{\partial f}{\Delta\xi_2}(1+g^2)\right)\frac{\mathrm{Im}g}{|g|^2+1} \\
&\quad -\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g\right)\frac{|g|^2-1}{|g|^2+1}, \\
I_{22} &= \mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}(g+g^{\sigma_2})\right)\frac{\mathrm{Re}g}{|g|^2+1} + \mathrm{Im}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}(g+g^{\sigma_2})\right)\frac{\mathrm{Im}g}{|g|^2+1} \\
&\quad -\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)\frac{|g|^2-1}{|g|^2+1}.
\end{aligned}
$$

Then

$$
\frac{\partial^2 x}{\Delta\xi_2^2}\cdot N = I_{21}+I_{22}.
$$

For I_{21}, we have

$$
\begin{aligned}
I_{21} &= -\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}(1-g^2)\right)\frac{\mathrm{Re}g}{|g|^2+1} + \mathrm{Im}\left(\frac{\partial f}{\Delta\xi_2}(1+g^2)\right)\frac{\mathrm{Im}g}{|g|^2+1} \\
&\quad -\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g\right)\frac{|g|^2-1}{|g|^2+1} \\
&= -\frac{1}{|g|^2+1}\left(\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}+\frac{\partial f}{\Delta\xi_2}g^2\right)\mathrm{Re}g - \mathrm{Im}\left(\frac{\partial f}{\Delta\xi_2}-\frac{\partial f}{\Delta\xi_2}g^2\right)\mathrm{Im}g\right. \\
&\quad \left. -\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g\right)(|g|^2-1)\right) \\
&= -\frac{1}{|g|^2+1}\left(\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}\right)\mathrm{Re}g + \mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g^2\right)\mathrm{Re}g + \mathrm{Im}\left(\frac{\partial f}{\Delta\xi_2}\right)\mathrm{Im}g + \mathrm{Im}\left(\frac{\partial f}{\Delta\xi_2}g^2\right)\mathrm{Im}g\right. \\
&\quad \left. -\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g\right)(|g|^2-1)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{|g|^2+1}\left(\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g\right)+\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g^2\right)\mathrm{Re}g+\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_2}g^2\right)\mathrm{Im}g\right.\\
&\quad\left.-\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g\right)|g|^2+\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g\right)\right)\\
&= \frac{1}{|g|^2+1}\left(\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g^2\right)\mathrm{Re}g+\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_2}g^2\right)\mathrm{Im}g-\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g\right)|g|^2\right)\\
&= -\frac{1}{|g|^2+1}\left(-\left(\frac{\partial f}{\Delta\xi_2}g\right)(\mathrm{Re}g)^2+\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_2}g\right)\mathrm{Im}g\mathrm{Re}g-\mathrm{Im}\left(\frac{\partial f}{\Delta\xi_2}g\right)\mathrm{Re}g\mathrm{Im}g\right.\\
&\quad\left.-\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g\right)(\mathrm{Im}g)^2+\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g\right)(\mathrm{Re}g)^2+\mathrm{Re}\left(\frac{\partial f}{\Delta\xi_2}g\right)(\mathrm{Im}g)^2\right)\\
&= 0.
\end{aligned}
$$

Noiw, we consider I_{12}. We have

$$
\begin{aligned}
I_{22} &= \mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}(g+g^{\sigma_2})\right)\frac{\mathrm{Re}g}{|g|^2+1}+\mathrm{Im}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}(g+g^{\sigma_2})\right)\frac{\mathrm{Im}g}{|g|^2+1}\\
&\quad-\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)\frac{|g|^2-1}{|g|^2+1}\\
&= -\frac{1}{|g|^2+1}\left(-\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}(g+g^{\sigma_2})\right)\mathrm{Re}g-\mathrm{Im}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}(g+g^{\sigma_2})\right)\mathrm{Im}g\right.\\
&\quad\left.+\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)(|g|^2-1)\right)\\
&= -\frac{1}{|g|^2+1}\left(-\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)\mathrm{Re}(g+g^{\sigma_2})\mathrm{Re}g+\mathrm{Im}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)\mathrm{Im}(g+g^{\sigma_2})\mathrm{Re}g\right.\\
&\quad-\mathrm{Im}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)\mathrm{Re}(g+g^{\sigma_2})\mathrm{Im}g-\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)\mathrm{Im}(g+g^{\sigma_2})\mathrm{Im}g\\
&\quad\left.+\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)(\mathrm{Re}g)^2+\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)(\mathrm{Im}g)^2-\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)\right)\\
&= -\frac{1}{|g|^2+1}\left(-\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)(\mathrm{Re}g)^2-\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)\mathrm{Re}g^{\sigma_2}\mathrm{Re}g\right.\\
&\quad+\mathrm{Im}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)\mathrm{Im}g\mathrm{Re}g+\mathrm{Im}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)\mathrm{Im}g^{\sigma_2}\mathrm{Re}g
\end{aligned}
$$

$$-\mathrm{Im}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)\mathrm{Im}g\mathrm{Re}g-\mathrm{Im}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)\mathrm{Re}g^{\sigma_2}\mathrm{Im}g$$

$$-\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)(\mathrm{Im}g)^2-\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)\mathrm{Im}g^{\sigma_2}\mathrm{Im}g$$

$$+\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)(\mathrm{Re}g)^2+\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)(\mathrm{Im}g)^2-\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)\Bigg)$$

$$=-\frac{1}{|g|^2+1}\Bigg(-\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)(\mathrm{Re}g^{\sigma_2}\mathrm{Re}g+\mathrm{Im}g^{\sigma_2}\mathrm{Im}g)$$

$$+\mathrm{Im}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)(\mathrm{Im}g^{\sigma_2}\mathrm{Re}g-\mathrm{Re}g^{\sigma_2}\mathrm{Im}g)\Bigg),$$

i.e.,

$$I_{22}=-\frac{1}{|g|^2+1}\Bigg(-\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)(\mathrm{Re}g^{\sigma_2}\mathrm{Re}g+\mathrm{Im}g^{\sigma_2}\mathrm{Im}g)$$

$$+\mathrm{Im}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)(\mathrm{Im}g^{\sigma_2}\mathrm{Re}g-\mathrm{Re}g^{\sigma_2}\mathrm{Im}g)\Bigg).$$

Consequently

$$\frac{\partial^2 x}{\Delta\xi_2^2}\cdot N=I_{21}+I_{22}$$

$$=-\frac{1}{|g|^2+1}\Bigg(-\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)(\mathrm{Re}g^{\sigma_2}\mathrm{Re}g+\mathrm{Im}g^{\sigma_2}\mathrm{Im}g)$$

$$+\mathrm{Im}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)(\mathrm{Im}g^{\sigma_2}\mathrm{Re}g-\mathrm{Re}g^{\sigma_2}\mathrm{Im}g)\Bigg).$$

Now, since

$$\frac{\partial}{\Delta\xi_2}\left(\frac{\partial x_1}{\Delta\xi_2}^{\sigma_1}\right)=-\frac{\partial}{\Delta\xi_2}(\mathrm{Re}\phi_1),$$

$$\frac{\partial}{\Delta\xi_2}\left(\frac{\partial x_2}{\Delta\xi_2}^{\sigma_1}\right)=-\frac{\partial}{\Delta\xi_2}(\mathrm{Re}\phi_2),$$

$$\frac{\partial}{\Delta\xi_2}\left(\frac{\partial x_3}{\Delta\xi_2}^{\sigma_1}\right)=-\frac{\partial}{\Delta\xi_2}(\mathrm{Re}\phi_3),$$

we obtain

$$\frac{\partial}{\Delta\xi_2}\left(\frac{\partial x}{\Delta\xi_2}^{\sigma_1}\right)\cdot N=-\frac{1}{|g|^2+1}\Bigg(-\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)(\mathrm{Re}g^{\sigma_2}\mathrm{Re}g+\mathrm{Im}g^{\sigma_2}\mathrm{Im}g)$$

$$+\mathrm{Im}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)(\mathrm{Im}g^{\sigma_2}\mathrm{Re}g-\mathrm{Re}g^{\sigma_2}\mathrm{Im}g)\Bigg).$$

Because

$$\frac{\partial}{\Delta\xi_1}\left(\frac{\partial x_1}{\Delta\xi_2}^{\sigma_1}\right)=-\frac{\partial}{\Delta\xi_1}(\mathrm{Re}\phi_1),$$

$$\frac{\partial}{\Delta\xi_1}\left(\frac{\partial x_2}{\Delta\xi_2}^{\sigma_1}\right)=-\frac{\partial}{\Delta\xi_1}(\mathrm{Re}\phi_2),$$

$$\frac{\partial}{\Delta\xi_1}\left(\frac{\partial x_3}{\Delta\xi_2}^{\sigma_1}\right)=-\frac{\partial}{\Delta\xi_1}(\mathrm{Re}\phi_3),$$

we find that

$$\frac{\partial}{\Delta\xi_1}\left(\frac{\partial x}{\Delta\xi_2}^{\sigma_1}\right)=-\frac{1}{|g|^2+1}\Bigg(-\mathrm{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)(\mathrm{Re}g^{\sigma_1}\mathrm{Re}g+\mathrm{Im}g^{\sigma_1}\mathrm{Im}g)$$

$$+\mathrm{Im}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)(\mathrm{Im}g^{\sigma_1}\mathrm{Re}g-\mathrm{Re}g^{\sigma_1}\mathrm{Im}g)\Bigg).$$

Set

$$H_1(f,g)=-\mathrm{Re}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)(\mathrm{Re}g^{\sigma_1}\mathrm{Re}g+\mathrm{Im}g^{\sigma_1}\mathrm{Im}g)$$

$$+\mathrm{Im}\left(f^{\sigma_1}\frac{\partial g}{\Delta\xi_1}\right)(\mathrm{Im}g^{\sigma_1}\mathrm{Re}g-\mathrm{Re}g^{\sigma_1}\mathrm{Im}g),$$

$$H_2(f,g)=\mathrm{Re}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)(\mathrm{Re}g^{\sigma_2}\mathrm{Re}g-\mathrm{Im}g^{\sigma_2}\mathrm{Im}g)$$

$$-\mathrm{Im}\left(f^{\sigma_2}\frac{\partial g}{\Delta\xi_2}\right)(\mathrm{Im}g^{\sigma_2}\mathrm{Re}g+\mathrm{Re}g^{\sigma_2}\mathrm{Im}g).$$

Therefore

$$\frac{\partial^2 x}{\Delta\xi_1^2}\cdot N=\frac{1}{|g|^2+1}H_1(f,g),$$

$$\frac{\partial}{\Delta\xi_2}\left(\frac{\partial x}{\Delta\xi_2}^{\sigma_1}\right)=\frac{1}{|g|^2+1}H_2(f,g),$$

$$\frac{\partial}{\Delta\xi_1}\left(\frac{\partial x}{\Delta\xi_2}\right)^{\sigma_1}=-\frac{1}{|g|^2+1}H_1(f,g),$$

$$\frac{\partial^2 x^{\sigma_1}}{\Delta\xi_2^2} = \frac{1}{|g|^2+1} H_2(f,g).$$

From here, for the second fundamental form we obtain the following representation

$$\begin{aligned}\sum_{i,j=1}^{2} b_{ij}(N)\frac{d\xi_i}{\Delta t}\frac{\Delta\xi_j}{\Delta t} &= \frac{1}{|g|^2+1}H_1(f,g)\left(\frac{d\xi_1}{\Delta t}\right)^2 - \frac{1}{|g|^2+1}H_1(f,g)\frac{d\xi_1}{\Delta t}\frac{d\xi_2}{\Delta t}\\ &\quad + \frac{1}{|g|^2+1}H_2(f,g)\frac{d\xi_1}{\Delta t}\frac{d\xi_2}{\Delta t} - \frac{1}{|g|^2+1}H_2(f,g)\left(\frac{d\xi_2}{\Delta t}\right)^2\\ &= \frac{1}{|g|^2+1}\Bigg(H_1(f,g)\left(\frac{d\xi_1}{\Delta t}\right)^2 - (H_1(f,g)-H_2(f,g))\frac{d\xi_1}{\Delta t}\frac{d\xi_2}{\Delta t}\\ &\quad - H_2(f,g)\left(\frac{d\xi_2}{\Delta t}\right)^2\Bigg).\end{aligned}$$

Theorem 3.11. *Suppose that*

$$H_1(f,g) = -H_2(f,g)$$

and

$$L(f,g) = \frac{1}{|g|^2+1}H(f,g)(1+i).$$

Then, for the normal curvature we have the representation

$$k(N) = \frac{4}{(|f|(1+|g|^2))^2} Re\left(L(f,g)e^{2i\alpha}\right),$$

where

$$\frac{d\zeta}{\Delta t} = \left|\frac{d\zeta}{\Delta t}\right| e^{i\alpha},$$

$$\zeta = \xi_1 + i\xi_2.$$

The principal curvatures are given by

$$\begin{aligned} k_1(N) &= \frac{4|L(f,g)|}{(|f|(1+|g|^2))^2},\\ k_2(N) &= -\frac{4|L(f,g)|}{(|f|(1+|g|^2))^2}. \end{aligned} \tag{3.11}$$

Proof. Note that

$$L(f,g)\left(\frac{d\zeta}{\Delta t}\right) = \frac{1}{|g|^2+1}H(f,g)(1+i)\left(\frac{d\xi_1}{\Delta t}+i\frac{d\xi_2}{\Delta t}\right)^2$$
$$= \frac{1}{|g|^2+1}H(f,g)(1+i)\left(\left(\frac{d\xi_1}{\Delta t}\right)^2-\left(\frac{d\xi_2}{\Delta t}\right)^2+2i\frac{d\xi_1}{\Delta t}\frac{d\xi_2}{\Delta t}\right),$$

whereupon

$$\mathrm{Re}\left(L(f,g)\left(\frac{d\zeta}{\Delta t}\right)^2\right) = \frac{1}{|g|^2+1}H(f,g)\left(\left(\frac{d\xi_1}{\Delta t}\right)^2-\left(\frac{d\xi_2}{\Delta t}\right)^2\right)-\frac{2}{|g|^2+1}H(f,g)\frac{d\xi_1}{\Delta t}\frac{d\xi_2}{\Delta t}$$
$$= \frac{1}{|g|^2+1}H(f,g)\left(\left(\frac{d\xi_1}{\Delta t}\right)^2+\left(\frac{d\xi_1}{\Delta t}\right)^2-2\frac{d\xi_1}{\Delta t}\frac{d\xi_2}{\Delta t}\right).$$

Therefore

$$\sum_{i,j=1}^{2} b_{ij}(N)\frac{d\xi_i}{\Delta t}\frac{d\xi_j}{\Delta t} = \frac{1}{|g|^2+1}H(f,g)\left(\left(\frac{d\xi_1}{\Delta t}\right)^2+\left(\frac{d\xi_1}{\Delta t}\right)^2-2\frac{d\xi_1}{\Delta t}\frac{d\xi_2}{\Delta t}\right)$$
$$= \mathrm{Re}\left(L(f,g)\left(\frac{d\zeta}{\Delta t}\right)^2\right)$$
$$= \mathrm{Re}\left(L(f,g)\left|\frac{d\zeta}{\Delta t}\right|^2 e^{2i\alpha}\right)$$

and

$$k(N) = \frac{\sum_{i,j=1}^{2} b_{ij}(N)\frac{d\xi_i}{\Delta t}\frac{d\xi_j}{\Delta t}}{F_{111}\left(\frac{d\xi_1}{\Delta t}\right)^2+F_{122}\left(\frac{d\xi_2}{\Delta t}\right)^2}$$
$$= \frac{\mathrm{Re}\left(L(f,g)\left|\frac{d\zeta}{\Delta t}\right|^2 e^{2i\alpha}\right)}{\left(\frac{|f|(1+|g|^2)}{2}\right)^2\left|\frac{d\zeta}{\Delta t}\right|^2}$$
$$= \frac{4}{(|f|(1+|g|^2))^2}\mathrm{Re}\left(L(f,g)e^{2i\alpha}\right).$$

The maximum and minimum of this expression, as α varies from 0 to 2π, were defined to be the principal curvatures. Then, we have (3.11). This completes the proof.

Definition 3.27. The Gauss curvature K at a point is defined to be the product of the principal curvatures.

Corollary 3.1. *Suppose that all conditions of Theorem 3.11 hold. Then*

$$K = -\frac{16|L(f,g)|^2}{|f|^2(1+|g|^2)^4}.$$

Consider, for an arbitrary minimal surface in $\mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \times \mathbb{T}_{(3)}$, the following sequence of mappings

$$U \xrightarrow{x(\zeta)} S \overset{\text{Gauss map}}{\longrightarrow} \Sigma \overset{\text{stereographic projection}}{\longrightarrow} w-\text{plane}, \qquad (3.12)$$

where Σ is the unit sphere. By Theorem 3.6, it follows that the composed map is

$$g(\zeta) : U \to w-\text{plane}.$$

Consider an arbitrary differentiable curve $\zeta(t)$ in U and its image under each of the maps in (3.12). With $s(t)$ we will denote the arclength of the image on S. Then

$$\frac{ds}{\Delta t} = \frac{1}{2}|f|(1+|g|^2)\left|\frac{d\zeta}{\Delta t}\right|.$$

The arclength of the image in the w-plane is

$$\left|\frac{dw}{\Delta t}\right| = |L(f,g)|\left|\frac{d\zeta}{\Delta t}\right|.$$

Let $\psi(t)$ denote the arclength of the sphere. Then, by the definition for stereographic projection, it follows that

$$\frac{d\psi}{\Delta t} = \frac{2}{1+|w|^2}\left|\frac{dw}{\Delta t}\right|.$$

Hence,

$$\begin{aligned}
\frac{\frac{d\psi}{\Delta t}}{\frac{ds}{\Delta t}} &= \frac{\frac{2}{1+|w|^2}\left|\frac{dw}{\Delta t}\right|}{\frac{1}{2}|f|(1+|g|^2)\left|\frac{d\zeta}{\Delta t}\right|} \\
&= \frac{\frac{2}{1+|g|^2}|L(f,g)|\left|\frac{d\zeta}{\Delta t}\right|}{\frac{1}{2}|f|(1+|g|^2)\left|\frac{d\zeta}{\Delta t}\right|} \\
&= \frac{4|L(f,g)|}{|f|(1+|g|^2)^2} \\
&= \sqrt{|K|}.
\end{aligned}$$

Let now, U_1 be a domain whose closue is in U. The surface defined by the restriction of $x(\zeta)$ to U_1 has total curvature given by

$$\begin{aligned}
\iint\limits_{U_1} K\Delta A &= \iint\limits_{U_1} K\lambda^2 \Delta\xi_1 \Delta\xi_2 \\
&= -\iint\limits_{U_1} \frac{16|L(f,g)|^2}{|f|^2(1+|g|^2)^4}\,\frac{|f|^2(1+|g|^2)^2}{4}\Delta\xi_1 \Delta\xi_2 \\
&= -\iint\limits_{U_1} \frac{4|L(f,g)|^2}{(1+|g|^2)^2}\Delta\xi_1 \Delta\xi_2 \\
&= -\iint\limits_{U_1} \left(\frac{2|L(f,g)|}{1+|g|^2}\right)^2 \Delta\xi_1 \Delta\xi_2,
\end{aligned}$$

which is the negative of the area of the image of U_1 under the Gauss map.

Chapter 4
The Variational Approach

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be time scales with forward jump operators and delta differentiation operators σ_1, σ_2 and Δ_1, Δ_2, respectively. With $\mathscr{C}_{rd}$ we denote the set of functions $f(x,y)$ on $\mathbb{T}_1 \times \mathbb{T}_2$ with the following properties.

1. f is rd-continuous in x for fixed y.
2. f is rd-continuous in y for fixed x.
3. If $(x_0,y_0) \in \mathbb{T}_1 \times \mathbb{T}_2$ with x_0 right-dense or maximal and y_0 right-dense or maximal, then f is continuous at (x_0,y_0).
4. If x_0 and y_0 are both left-sided, then the limit $f(x,y)$ exists(finite) as (x,y) approaches (x_0,y_0) along any path in

$$\{(x,y) \in \mathbb{T}_1 \times \mathbb{T}_2 : x < x_0, \quad y < y_0\}.$$

By $\mathscr{C}_{rd}^{(1)}$ we denote the set of all continuous functions for which both the Δ_1-partial derivative and the Δ_2-partial derivative exist and are of the class C_{rd}.

4.1 Statement of the Variational Problem

Let $E \subset \mathbb{T}_1 \times \mathbb{T}_2$ be a set of type ω and let Γ be its positively oriented fence. Suppose that a function

$$L(x,y,u,p,q), \quad (x,y) \in E \bigcup \Gamma \quad and \quad (u,p,q) \in \mathbb{R}^3,$$

is given, it is continuous together with its partial delta derivatives of the first and second order with respect to x, y and partial usual derivatives of the first and second order with respect to u, p, q. Consider the functional

$$\mathscr{L}(u) = \int\int_E L(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\Delta_1 x \Delta_2 y \quad (4.1)$$

whose domain of definition $D(\mathscr{L})$ consists of functions $u \in \mathscr{C}_{rd}^{(1)}(E \bigcup \Gamma)$ satisfying the "boundary conditions"

$$u = g(x,y) \quad on \quad \Gamma, \tag{4.2}$$

where g is a fixed function defined and continuous on Γ.

Definition 4.1. We call functions $u \in D(\mathscr{L})$ admissible.

Definition 4.2. The functions $\eta \in \mathscr{C}_{rd}^{(1)}(E \bigcup \Gamma)$ and $\eta = 0$ on Γ, are called admissible variations.

If $f \in \mathscr{C}_{rd}^{(1)}(E \bigcup \Gamma)$, we define the norm

$$\|f\| = \sup_{(x,y)\in E\bigcup\Gamma} |f(x,y)| + \sup_{(x,y)\in E} \left| f^{\Delta_1}(x,\sigma_2(y)) \right| + \sup_{(x,y)\in E} \left| f^{\Delta_2}(\sigma_1(x),y) \right|.$$

Definition 4.3. A function $\hat{u} \in D(\mathscr{L})$ is called a weak local minimum of $\mathscr{L}$ provided there exists a $\delta > 0$ such that

$$\mathscr{L}(\hat{u}) \le \mathscr{L}(u)$$

for all $u \in D(\mathscr{L})$ with

$$\|u - \hat{u}\| < \delta.$$

If

$$\mathscr{L}(\hat{u}) < \mathscr{L}(u)$$

for all such $u \neq \hat{u}$, then $\hat{u}$ is said to be proper weak local minimum.

4.2 First and Second Variation

For a fixed element $u \in D(\mathscr{L})$ and a fixed admissible variation η, we define $\Phi : \mathbb{R} \to \mathbb{R}$ as follows.

$$\Phi(\varepsilon) = \mathscr{L}(u + \varepsilon\eta).$$

Definition 4.4. The first and second variation $\mathscr{L}$ at the point u are defined by

$$\mathscr{L}_1(u,\eta) = \Phi'(0) \quad and \quad \mathscr{L}_2(u,\eta) = \Phi''(0),$$

respectively.

Theorem 4.1 (Necessary Condition). *If $\hat{u} \in D(\mathscr{L})$ is a local minimum of $\mathscr{L}$, then*

$$\mathscr{L}_1(u,\eta) = 0 \quad \textit{and} \quad \mathscr{L}_2(u,\eta) \geq 0$$

for all admissible variations η.

Proof. Assume that $\mathscr{L}$ has a local minimum at $\hat{u} \in D(\mathscr{L})$. Let η be an arbitrary admissible variation. Then

$$\Phi'(0) = \mathscr{L}_1(\hat{u},\eta) \quad \textit{and} \quad \Phi''(0) = \mathscr{L}_2(\hat{u},\eta).$$

By Taylor's formula, we get

$$\Phi(\varepsilon) = \Phi(0) + \Phi'(0)\varepsilon + \frac{1}{2!}\Phi''(\alpha)\varepsilon^2,$$

where $|\alpha| \in (0,|\varepsilon|)$. If $|\varepsilon|$ is sufficiently small, then

$$\|\hat{u} + \varepsilon\eta - \hat{u}\| = |\varepsilon|\|\eta\|$$

will be as small as we please. Hence, from the definition of a local minimum, we obtain

$$\mathscr{L}(\hat{u} + \varepsilon\eta) \geq \mathscr{L}(\hat{u}),$$

i.e.,

$$\Phi(\varepsilon) \geq \Phi(0).$$

Therefore Φ has a local minimum for $\varepsilon = 0$. From here,

$$\Phi'(0) = 0,$$

or, equivalently,

$$\mathscr{L}_1(\hat{u},\eta) = 0.$$

Since $\Phi'(0) = 0$, we have

$$\Phi(\varepsilon) - \Phi(0) = \frac{1}{2}\Phi''(\alpha)\varepsilon^2.$$

Therefore $\Phi''(\alpha) \geq 0$ for all ε whose absolute values is sufficiently small. Letting $\varepsilon \to 0$ and using that $\alpha \to \infty$, as $\varepsilon \to 0$, and Φ'' is continuous, we get

$$\Phi''(0) \geq 0,$$

or, equivalently,

$$\mathscr{L}_2(\hat{u},\eta) \geq 0.$$

This completes the proof.

Theorem 4.2 (Sufficient Condition). *Let $\hat{u} \in D(\mathscr{L})$ be such that*

$$\mathscr{L}_1(\hat{u}, \eta) = 0$$

for all admissible variations η. If $\mathscr{L}_2(u, \eta) \le 0$ for all $u \in D(\mathscr{L})$ and all admissible variations η, then $\mathscr{L}$ has an absolute minimum at the point $\hat{u}$. If $\mathscr{L}_2(u, \eta) \le 0$ for all u in some neighbourhood of the point $\hat{u}$ and all admissible variations η, then the functional $\mathscr{L}$ has a local maximum at $\hat{u}$.

Proof. For the function Φ we have

$$\Phi(1) = \Phi(0) + \Phi'(0) + \frac{1}{2!}\Phi''(\alpha), \quad \alpha \in (0,1). \tag{4.3}$$

Note that

$$\begin{aligned}
\Phi(1) &= \mathscr{L}(\hat{u} + \eta), \quad \Phi(0) = \mathscr{L}(\hat{u}),\\
\Phi'(0) &= \mathscr{L}_1(\hat{u}, \eta)\\
&= 0,\\
\Phi''(\alpha) &= \left(\frac{d^2}{d\varepsilon^2}\mathscr{L}(\hat{u} + \varepsilon\eta)\right)\Big|_{\varepsilon=\alpha}\\
&= \left(\frac{d^2}{d\beta^2}\mathscr{L}(\hat{u} + \alpha\eta + \beta\eta)\right)\Big|_{\beta=0}\\
&= \mathscr{L}_2(\hat{u} + \alpha\eta, \eta).
\end{aligned}$$

Hence and (4.3), we obtain

$$\mathscr{L}(\hat{u} + \eta) = \mathscr{L}(\hat{u}) + \frac{1}{2!}\mathscr{L}_2(\hat{u} + \alpha\eta, \eta)$$

for all admissible variations η, where $\alpha \in (0,1)$. Suppose that $\mathscr{L}_2(u, \eta) \ge 0$ for all $u \in D(\mathscr{L})$ and all admissible variations η. If $u \in D(\mathscr{L})$, then putting

$$\eta = u - \hat{u},$$

we get

$$\mathscr{L}(u) \ge \mathscr{L}(\hat{u}).$$

Then $\mathscr{L}$ has an absolute minimum at the point $\hat{u}$. Now we suppose that $\mathscr{L}_2(u, \eta) \le 0$ for all u in some neighbourhood of the point $\hat{u}$ and all admissible variations η. There exists $r > 0$ such that for $u \in D(\mathscr{L})$ and

$$\|u-\hat{u}\| < r,$$

we have $\mathscr{L}_2(u,\eta) \leq 0$ for all admissible variations η. We take such an element u and we put $\eta = u - \hat{u}$. Then

$$\mathscr{L}(u) = \mathscr{L}(\hat{u}) + \frac{1}{2}\mathscr{L}_2(\hat{u}+\alpha\eta,\eta).$$

Note that

$$\begin{aligned}\|\hat{u}+\alpha\eta-\hat{u}\| &= \|\alpha\eta\| \\ &= |\alpha|\|\eta\| \\ &\leq \|\eta\| \\ &= \|u-\hat{u}\| \\ &< r.\end{aligned}$$

Hence,

$$\mathscr{L}_2(\hat{u}+\alpha\eta,\eta) \leq 0,$$

and, then

$$\mathscr{L}(u) \leq \mathscr{L}(\hat{u}).$$

This completes the proof.

By Theorem 4.1 and Theorem 4.2, it follows that

$$\begin{aligned}\mathscr{L}_1(u,\eta) = \int\int_E \Big(& L_u(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\eta(\sigma_1(x),\sigma_2(y)) \\ & + L_p(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\eta^{\Delta_1}(x,\sigma_2(y)) \\ & + L_q(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\eta^{\delta_2}(\sigma_1(x),y)\Big)\Delta_1 x\Delta_2 y,\end{aligned} \tag{4.4}$$

and

$$\mathscr{L}_2(u,\eta) = \int\int_E \Bigg(L_{uu}(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\,(\eta(\sigma_1(x),\sigma_2(y)))^2$$

$$+L_{pp}(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\left(\eta^{\Delta_1}(x,\sigma_2(y))\right)^2$$

$$+L_{qq}(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\left(\eta^{\Delta_2}(\sigma_1(x),y)\right)^2$$

$$+2L_{up}(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\eta(\sigma_1(x),\sigma_2(y))\eta^{\Delta_1}(x,\sigma_2(y))$$

$$+2L_{uq}(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\eta(\sigma_1(x),\sigma_2(y))\eta^{\Delta_2}(\sigma_1(x),y)$$

$$+2L_{pq}(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\eta^{\Delta_1}(x,\sigma_2(y))\eta^{\Delta_2}(\sigma_1(x),y)\Bigg)\Delta_1 x\Delta_2 y. \tag{4.5}$$

Example 4.1. Let

$$L(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))$$

$$= x+y+u(\sigma_1(x),\sigma_2(y))+\left(u^{\Delta_1}(x,\sigma_2(y))\right)^2$$

$$+\left(u^{\Delta_2}(\sigma_1(x),y)\right)^3.$$

Here

$$L(x,y,u,p,q) = x+y+u+p^2+q^3.$$

Then

$$L_u(x,y,u,p,q) = 1,$$

$$L_p(x,y,u,p,q) = 2p,$$

$$L_q(x,y,u,p,q) = 3q^2,$$

$$L_{uu}(x,y,u,p,q) = 0,$$

$$L_{pp}(x,y,u,p,q) = 2,$$

$$L_{qq}(x,y,u,p,q) = 6q,$$

$$L_{uq}(x,y,u,p,q) = 0,$$

$$L_{up}(x,y,u,p,q) = 0,$$

$$L_{pq}(x,y,u,p,q) = 0.$$

Therefore the equations (4.4) and (4.5) take the form

$$\mathscr{L}_1(u,\eta) = \int\int_E \Bigg(\eta(\sigma_1(x),\sigma_2(y)) + 2u^{\Delta_1}(x,\sigma_2(y))\eta^{\Delta_1}(x,\sigma_2(y)) + 3\left(u^{\Delta_1}(\sigma_1(x),y)\right)^2 \eta^{\Delta_2}(\sigma_1(x),y) \Bigg) \Delta_1 x \Delta_2 y,$$

$$\mathscr{L}_2(u,\eta) = \int\int_E \left(2\left(\eta^{\Delta_1}(x,\sigma_2(y))\right)^2 + 6u^{\Delta_2}(\sigma_1(x),y)\left(\eta^{\Delta_2}(\sigma_1(x),y)\right)^2 \right) \Delta_1 x \Delta_2 y.$$

4.3 Euler's Condition

Let E is an ω-type subset of $\mathbb{T}_1 \times \mathbb{T}_2$ and Γ be the positively oriented fence of E. We set

$$E^\sigma = \{(x,y) \in E : (\sigma_1(x),\sigma_2(y)) \in E\}.$$

Lemma 4.1 (Dubois-Reymond's Lemma). *If $M(x,y)$ is continuous on $E\bigcup\Gamma$ with*

$$\int\int_E M(x,y)\eta(\sigma_1(x),\sigma_2(y))\Delta_1 x \Delta_2 y = 0$$

for every admissible variation η, then

$$M(x,y) = 0 \quad for \quad all \quad (x,y) \in E^\sigma.$$

Proof. Assume the contrary. Without loss of generality, we suppose that $(x_0,y_0) \in E^\sigma$ is such that $M(x_0,y_0) > 0$. The continuity of $M(x,y)$ ensures that $M(x,y)$ is positive in a rectangle

$$\Omega = [x_0,x_1) \times [y_0,y_1) \subset E$$

for some points $x_1 \in \mathbb{T}_1$, $y_1 \in \mathbb{T}_2$ such that

$$\sigma_1(x_0) \le x_1 \quad and \quad \sigma_2(y_0) \le y_1.$$

We set

$$\eta(x,y) = \begin{cases} (x-x_0)^2(x-\sigma_1(x_1))^2(y-y_0)(y-\sigma_2(y_1))^2 \quad for \quad (x,y) \in \Omega \\ 0 \quad for \quad (x,y) \in E \backslash \Omega. \end{cases}$$

We have that $\eta \in \mathscr{C}_{rd}^{(1)}(E \bigcup \Gamma)$, $\eta\Big|_{\Gamma} = 0$, i.e., η is an admissible variation. We have that

$$\int\int_E M(x,y)\eta(\sigma_1(x),\sigma_2(y))\Delta_1 x \Delta_2 y = \int\int_\Omega M(x,y)\eta(\sigma_1(x),\sigma_2(y))\Delta_1 x \Delta_2 y$$
$$> 0,$$

which is a contradiction. This completes the proof.

Theorem 4.3 (Euler's Necessary Condition). *Suppose that an admissible function $\hat{u}$ provides a local minimum for $\mathscr{L}$ and the function $\hat{u}$ has continuous partial delta derivatives of the second order. Then $\hat{u}$ satisfies the Euler-Lagrange equation*

$$\begin{aligned} 0 = & L_u(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y)) \\ & -L_p^{\Delta_1}(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y)) \\ & -L_q^{\Delta_2}(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y)) \end{aligned} \tag{4.6}$$

for $(x,y) \in E^\sigma$.

Proof. Since $\hat{u}$ is a local minimum for $\mathscr{L}$, by Theorem 4.1, it follows that

$$\mathscr{L}_1(\hat{u},\eta) = 0$$

for all admissible variations η. Hence and (4.4), applying integration by parts and Green's formula, we get

$$\begin{aligned} 0 = & \mathscr{L}_1(\hat{u},\eta) \\ = & \int\int_E \Bigg(L_u(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y)) \\ & \times\eta(\sigma_1(x),\sigma_2(y))\Delta_1 x\Delta_2 y \\ & +L_p(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y)) \\ & \times\eta^{\Delta_1}(x,\sigma_2(y)) \end{aligned}$$

$$+L_q(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\eta^{\Delta_2}(\sigma_1(x),y)\Bigg)\Delta_1 x\Delta_2 y$$

$$= \int\int_E L_u(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))$$

$$\times\eta(\sigma_1(x),\sigma_2(y))\Delta_1 x\Delta_2 y$$

$$+\int\int_E\Bigg(L_p(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))$$

$$\times\eta^{\Delta_1}(x,\sigma_2(y))$$

$$+L_q(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))$$

$$\times\eta^{\Delta_2}(\sigma_1(x),y)\Delta_1 x\Delta_2 y$$

$$= \int\int_E L_u(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))$$

$$\times\eta(\sigma_1(x),\sigma_2(y))\Delta_1 x\Delta_2 y$$

$$+\int\int_E\Bigg(\frac{\partial}{\Delta_1 x}\Bigg(L_p(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))$$

$$\times\eta(x,\sigma_2(y))\Bigg)$$

$$+\frac{\partial}{\Delta_2 y}\Big(L_q(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))$$

$$\times\eta(\sigma_1(x),y)\Bigg)\Bigg)\Delta_1 x\Delta_2 y$$

$$-\int\int_E\Bigg(L_p^{\Delta_1}(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))$$

$$+L_q^{\Delta_2}(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\Bigg)$$

$$
\begin{aligned}
&\times\eta(\sigma_1(x),\sigma_2(y))\Delta_1 x\Delta_2 y\\
=&\int\int_E L_u(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\\
&\times\eta(\sigma_1(x),\sigma_2(y))\Delta_1 x\Delta_2 y\\
&-\int\int_E\left(\frac{\partial}{\Delta_1 x}\left(L_p^{\Delta_1}(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\right.\right.\\
&\left.+L_q^{\Delta_2}(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\right)\\
&\left.\left.\times\eta(\sigma_1(x),\sigma_2(y))\right)\right)\Delta_1 x\Delta_2 y\\
&+\int_\Gamma\left(L_p^{\Delta_1}(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\right.\\
&\times\eta(x,\sigma_2(y))\Delta_2 y\\
&-L_q^{\Delta_2}(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\\
&\left.\times\eta(\sigma_1(x),y)\Delta_1 x\right)\\
=&\int\int_E L_u(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\\
&\times\eta(\sigma_1(x),\sigma_2(y))\Delta_1 x\Delta_2 y\\
&-\int\int_E\left(L_p^{\Delta_1}(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\right.\\
&\left.-L_q^{\Delta_2}(x,y,u(\sigma_1(x),\sigma_2(y)),u^{\Delta_1}(x,\sigma_2(y)),u^{\Delta_2}(\sigma_1(x),y))\right)\\
&\times\eta(\sigma_1(x),y)\Delta_1 x\Delta_2 y.
\end{aligned}
$$

From here and from Lemma 4.1, we get (4.6). This completes the proof.

Example 4.2. Let $\mathbb{T}_1=2^{\mathbb{N}_0}$, $\mathbb{T}_2=3^{\mathbb{N}_0}$. Consider the variational problem

$$\mathscr{L}(y) = \int\int_E x^2 y^3 u(2x,3y) u^{\Delta_1}(x,3y) u^{\Delta_2}(2x,y) \Delta_1 x \Delta_2 y \longrightarrow \min,$$

where

$$E = \{(x,y) \in \mathbb{T}_1 \times \mathbb{T}_2 : 1 \le x \le 8, \quad 1 \le y \le 27\}.$$

Here

$$L(x,y,u,p,q) = x^2 y^3 upq,$$

$$\sigma_1(x) = 2x, \quad x \in \mathbb{T}_1,$$

$$\sigma_2(y) = 3y, \quad y \in \mathbb{T}_2.$$

Then

$$\begin{aligned}
L_u(x,y,u,p,q) &= x^2 y^3 pq, \\
L_p(x,y,u,p,q) &= x^2 y^3 uq, \\
L_q(x,y,u,p,q) &= x^2 y^3 up, \\
L_p^{\Delta_1}(x,y,u,p,q) &= (\sigma_1(x) + x) y^3 uq \\
&= (2x + x) y^3 uq \\
&= 3xy^3 uq, \\
L_q^{\Delta_2}(x,y,u,p,q) &= x^2 \left((\sigma_2(y))^2 + y\sigma_2(y) + y^2 \right) up \\
&= x^2 \left((3y)^2 + y(3y) + y^2 \right) up \\
&= x^2 (9y^2 + 3y^2 + y^2) up \\
&= 13x^2 y^2 up.
\end{aligned}$$

The Euler-Lagrange equation takes the form

$$x^2 y^3 u^{\Delta_1}(x,3y) u^{\Delta_2}(2x,y) - 3xy^3 u(2x,3y) u^{\Delta_2}(2x,y)$$

$$-13x^2 y^2 u(2x,3y) u^{\Delta_1}(x,3y) = 0, \quad (x,y) \in E.$$

4.4 The Area of Parameterized Surfaces

Definition 4.5. Suppose that S is a σ_1-regular surface given by $f = f(t)$, $t \in U$. The σ_1-area of S is given by

$$A_{\sigma_1}(S) = \int\int_U |f_{t_1}^{\Delta_1} \times f_{t_2}^{\Delta_2\sigma_1}| \Delta_1 t_1 \Delta_2 t_2.$$

Suppose that S is a σ_1-regular surface defined by

$$f = f(t_1,t_2) = (f_1(t_1,t_2), f_2(t_1,t_2), f_3(t_1,t_2)), \quad (t_1,t_2) \in U.$$

Then

$$f_{t_1}^{\Delta_1} = \left(f_{1t_1}^{\Delta_1}, f_{2t_1}^{\Delta_1}, f_{3t_1}^{\Delta_1}\right),$$

$$f_{t_2}^{\Delta_2\sigma_1} = \left(f_{1t_2}^{\Delta_2\sigma_1}, f_{2t_2}^{\Delta_2\sigma_1}, f_{3t_2}^{\Delta_2\sigma_1}\right)$$

and

$$f_{t_1}^{\Delta_1} \times f_{t_2}^{\Delta_2\sigma_1} = \Big(f_{2t_1}^{\Delta_1} f_{3t_2}^{\Delta_2\sigma_1} - f_{2t_2}^{\Delta_2\sigma_1} f_{3t_1}^{\Delta_1}, f_{1t_2}^{\Delta_2\sigma_1} f_{3t_1}^{\Delta_1} - f_{1t_1}^{\Delta_1} f_{3t_2}^{\Delta_2\sigma_1},$$

$$f_{1t_1}^{\Delta_1} f_{2t_2}^{\Delta_2\sigma_1} - f_{1t_2}^{\Delta_2\sigma_1} f_{2t_1}^{\Delta_1}\Big).$$

Hence,

$$\begin{aligned}
\left(f_{t_1}^{\Delta_1} \times f_{t_2}^{\Delta_2\sigma_1}\right)^2 &= \left(f_{2t_1}^{\Delta_1} f_{3t_2}^{\Delta_2\sigma_1} - f_{2t_2}^{\Delta_2\sigma_1} f_{3t_1}^{\Delta_1}\right)^2 + \left(f_{1t_2}^{\Delta_2\sigma_1} f_{3t_1}^{\Delta_1} - f_{1t_1}^{\Delta_1} f_{3t_2}^{\Delta_2\sigma_1}\right)^2 \\
&\quad + \left(f_{1t_1}^{\Delta_1} f_{2t_2}^{\Delta_2\sigma_1} - f_{1t_2}^{\Delta_2\sigma_1} f_{2t_1}^{\Delta_1}\right)^2 \\
&= \left(f_{2t_1}^{\Delta_1}\right)^2 \left(f_{3t_2}^{\Delta_2\sigma_1}\right)^2 - 2 f_{2t_1}^{\Delta_1} f_{2t_2}^{\Delta_2\sigma_1} f_{3t_1}^{\Delta_1} f_{3t_2}^{\Delta_2\sigma_1} + \left(f_{2t_2}^{\Delta_2\sigma_1}\right)^2 \left(f_{3t_1}^{\Delta_1}\right)^2 \\
&\quad + \left(f_{1t_2}^{\Delta_2\sigma_1}\right)^2 \left(f_{3t_1}^{\Delta_1}\right)^2 + \left(f_{1t_1}^{\Delta_1}\right)^2 \left(f_{3t_2}^{\Delta_2\sigma_1}\right)^2 - 2 f_{1t_1}^{\Delta_1} f_{1t_2}^{\Delta_2\sigma_1} f_{3t_1}^{\Delta_1} f_{3t_2}^{\Delta_2\sigma_1} \\
&\quad + \left(f_{1t_1}^{\Delta_1}\right)^2 \left(f_{2t_2}^{\Delta_2\sigma_1}\right)^2 + \left(f_{1t_2}^{\Delta_2\sigma_1}\right)^2 \left(f_{2t_1}^{\Delta_1}\right)^2 - 2 f_{1t_2}^{\Delta_2\sigma_1} f_{1t_1}^{\Delta_1} f_{2t_1}^{\Delta_1} f_{2t_2}^{\Delta_2\sigma_1} \\
&= \left(f_{1t_1}^{\Delta_1}\right)^2 \left(f_{1t_2}^{\Delta_2\sigma_1}\right)^2 + \left(f_{1t_1}^{\Delta_1}\right)^2 \left(f_{2t_2}^{\Delta_2\sigma_1}\right)^2 + \left(f_{1t_1}^{\Delta_1}\right)^2 \left(f_{3t_2}^{\Delta_2\sigma_1}\right)^2 \\
&\quad + \left(f_{2t_1}^{\Delta_1}\right)^2 \left(f_{1t_2}^{\Delta_2\sigma_1}\right)^2 + \left(f_{2t_1}^{\Delta_1}\right)^2 \left(f_{2t_2}^{\Delta_2\sigma_1}\right)^2 + \left(f_{2t_1}^{\Delta_1}\right)^2 \left(f_{3t_2}^{\Delta_2\sigma_1}\right)^2
\end{aligned}$$

$$
\begin{aligned}
&+\left(f_{3t_1}^{\Delta_1}\right)^2\left(f_{1t_2}^{\Delta_2\sigma_1}\right)^2+\left(f_{3t_1}^{\Delta_1}\right)^2\left(f_{2t_2}^{\Delta_2\sigma_1}\right)^2+\left(f_{3t_1}^{\Delta_1}\right)^2\left(f_{3t_2}^{\Delta_2\sigma_1}\right)^2\\
&-\left(f_{1t_1}^{\Delta_1}\right)^2\left(f_{1t_2}^{\Delta_2\sigma_1}\right)^2-\left(f_{2t_1}^{\Delta_1}\right)^2\left(f_{2t_2}^{\Delta_2\sigma_1}\right)^2-\left(f_{3t_1}^{\Delta_1}\right)^2\left(f_{3t_2}^{\Delta_2\sigma_1}\right)^2\\
&-2f_{2t_1}^{\Delta_1}f_{2t_2}^{\Delta_2\sigma_1}f_{3t_1}^{\Delta_1}f_{3t_2}^{\Delta_2\sigma_1}-2f_{1t_1}^{\Delta_1}f_{1t_2}^{\Delta_2\sigma_1}f_{3t_1}^{\Delta_1}f_{3t_2}^{\Delta_2\sigma_1}\\
&-2f_{1t_2}^{\Delta_2\sigma_1}f_{1t_1}^{\Delta_1}f_{2t_1}^{\Delta_1}f_{2t_2}^{\Delta_2\sigma_1}\\
=&\left(\left(f_{1t_1}^{\Delta_1}\right)^2+\left(f_{2t_1}^{\Delta_1}\right)^2+\left(f_{3t_1}^{\Delta_1}\right)^2\right)\left(\left(f_{1t_2}^{\Delta_2\sigma_1}\right)^2+\left(f_{2t_2}^{\Delta_2\sigma_1}\right)^2+\left(f_{3t_2}^{\Delta_2\sigma_1}\right)^2\right)\\
&-\left(f_{1t_1}^{\Delta_1}f_{1t_2}^{\Delta_2\sigma_1}+f_{2t_1}^{\Delta_1}f_{2t_2}^{\Delta_2\sigma_1}+f_{3t_1}^{\Delta_1}f_{3t_2}^{\Delta_2\sigma_1}\right)^2\\
=&F_{111}F_{122}-F_{112}^2.
\end{aligned}
$$

Therefore

$$A_{\sigma_1}(S)=\int\int_U\sqrt{F_{111}F_{122}-F_{112}^2}\Delta_1t_1\Delta_2t_2.$$

Example 4.3. Let $\mathbb{T}_1=\mathbb{T}_2=\mathbb{Z}$, $\mathbb{T}_{(1)}=\mathbb{T}_{(2)}=\mathbb{T}_{(3)}=\mathbb{R}$ and

$$U=\{(t_1,t_2)\in\mathbb{T}_1\times\mathbb{T}_2:\ 0\le t_1,t_2\le 3\}.$$

Consider a surface S with a local parameterization

$$f(t_1,t_2)=(t_1+2,t_2-2,t_1+t_2),\quad (t_1,t_2)\in U.$$

Here,

$$
\begin{aligned}
f_1(t_1,t_2)&=t_1+2,\\
f_2(t_1,t_2)&=t_2-2,\\
f_3(t_1,t_2)&=t_1+t_2,\quad (t_1,t_2)\in U.
\end{aligned}
$$

Then

$$
\begin{aligned}
f_{1t_1}^{\Delta_1}(t_1,t_2)&=1,\\
f_{1t_2}^{\Delta_2}(t_1,t_2)&=0,
\end{aligned}
$$

$$
\begin{aligned}
f_{1t_2}^{\Delta_2\sigma_1}(t_1,t_2) &= 1, \\
f_{2t_1}^{\Delta_1}(t_1,t_2) &= 0, \\
f_{2t_2}^{\Delta_2}(t_1,t_2) &= 1, \\
f_{2t_2}^{\Delta_2\sigma_1}(t_1,t_2) &= 1, \\
f_{3t_1}^{\Delta_1}(t_1,t_2) &= 1, \\
f_{3t_2}^{\Delta_2}(t_1,t_2) &= 1, \\
f_{3t_2}^{\Delta_2\sigma_1}(t_1,t_2) &= 1.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
F_{111} &= \left(f_{1t_1}^{\Delta_1}(t_1,t_2)\right)^2 + \left(f_{2t_1}^{\Delta_1}(t_1,t_2)\right)^2 + \left(f_{3t_1}^{\Delta_1}(t_1,t_2)\right)^2 \\
&= 2, \\
F_{112} &= f_{1t_1}^{\Delta_1}(t_1,t_2) f_{1t_2}^{\Delta_2\sigma_1}(t_1,t_2) + f_{2t_1}^{\Delta_1}(t_1,t_2) f_{2t_2}^{\Delta_2\sigma_1}(t_1,t_2) \\
&\quad + f_{3t_1}^{\Delta_1}(t_1,t_2) f_{3t_2}^{\Delta_2\sigma_1}(t_1,t_2) \\
&= 1, \\
F_{122} &= \left(f_{1t_2}^{\Delta_2\sigma_1}(t_1,t_2)\right)^2 + \left(f_{2t_2}^{\Delta_2\sigma_1}(t_1,t_2)\right)^2 + \left(f_{3t_2}^{\Delta_2\sigma_1}(t_1,t_2)\right)^2 \\
&= 2
\end{aligned}
$$

so that

$$
\begin{aligned}
F_{111}F_{122} - F_{112}^2 &= 2 \cdot 2 - 1 \\
&= 3
\end{aligned}
$$

and

$$A_{\sigma_1}(S) = \int\int_U \sqrt{3}\Delta_1 t_1 \Delta_2 t_2$$
$$= 9\sqrt{3}.$$

4.5 Minimal Surfaces

Let S be a σ_1-regular surface given by

$$f = f(t_1,t_2) = (t_1,t_2,f_3(t_1,t_2)), \quad (t_1,t_2) \in U.$$

Set

$$p = f_{3t_1}^{\Delta_1},$$

$$q = f_{3t_2}^{\Delta_2\sigma_1}$$

and

$$H(t_1,t_2,f_3,p,q) = \left(\left(1+\left(f_{3t_1}^{\Delta_1}\right)^2\right)\left(1+\left(f_{3t_2}^{\Delta_2\sigma_1}\right)^2\right) - \left(f_{3t_1}^{\Delta_1} f_{3t_2}^{\Delta_2\sigma_1}\right)^2\right)^{\frac{1}{2}}$$
$$= \left((1+p^2)(1+q^2) - p^2q^2\right)^{\frac{1}{2}}$$
$$= \left(1+p^2+q^2+p^2q^2-p^2q^2\right)^{\frac{1}{2}}$$
$$= \sqrt{1+p^2+q^2}, \quad (t_1,t_2) \in U.$$

By the study in the previous section, we have that

$$A_{\sigma_1}(S) = \int\int_U H(t_1,t_2,f_3,p,q)\Delta_1 t_1 \Delta_2 t_2.$$

Then

$$H_p(t_1,t_2,f_3,p,q) = \frac{p}{\sqrt{1+p^2+q^2}},$$
$$H_q(t_1,t_2,f_3,p,q) = \frac{q}{\sqrt{1+p^2+q^2}},$$
$$H_{f_3}(t_1,t_2,f_3,p,q) = 0.$$

Hence, using the Pötzsche chain rule, we find

$$H_p^{\Delta_1}(t_1,t_2,f_3,p,q)$$

$$= \frac{p^{\Delta_1}\sqrt{1+p^2+q^2} - p\left(\int\limits_0^1 \frac{1}{\sqrt{1+p^2+q^2++h\mu_1(p^{\Delta_1}(p+p^{\sigma_1})+q^{\Delta_1}(q+q^{\sigma_1}))}}dh\right)(p^{\Delta_1}(p+p^{\sigma_1})+q^{\Delta_1}(q+q^{\sigma_1}))}{\sqrt{1+p^2+q^2}\sqrt{1+p^{2\sigma_1}+q^{2\sigma_1}}}$$

and

$$H_q^{\Delta_2}(t_1,t_2,f_3,p,q)$$

$$= \frac{q^{\Delta_2}\sqrt{1+p^2+q^2} - q\left(\int\limits_0^1 \frac{1}{\sqrt{1+p^2+q^2++h\mu_2(p^{\Delta_2}(p+p^{\sigma_2})+q^{\Delta_2}(q+q^{\sigma_2}))}}dh\right)(p^{\Delta_2}(p+p^{\sigma_2})+q^{\Delta_2}(q+q^{\sigma_2}))}{\sqrt{1+p^2+q^2}\sqrt{1+p^{2\sigma_2}+q^{2\sigma_2}}}.$$

Thus, the Euler-Lagrange equation is as follows

$$0 = \frac{p^{\Delta_1}\sqrt{1+p^2+q^2} - p\left(\int\limits_0^1 \frac{1}{\sqrt{1+p^2+q^2++h\mu_1(p^{\Delta_1}(p+p^{\sigma_1})+q^{\Delta_1}(q+q^{\sigma_1}))}}dh\right)(p^{\Delta_1}(p+p^{\sigma_1})+q^{\Delta_1}(q+q^{\sigma_1}))}{\sqrt{1+p^2+q^2}\sqrt{1+p^{2\sigma_1}+q^{2\sigma_1}}}$$

$$+\frac{q^{\Delta_2}\sqrt{1+p^2+q^2} - q\left(\int\limits_0^1 \frac{1}{\sqrt{1+p^2+q^2++h\mu_2(p^{\Delta_2}(p+p^{\sigma_2})+q^{\Delta_2}(q+q^{\sigma_2}))}}dh\right)(p^{\Delta_2}(p+p^{\sigma_2})+q^{\Delta_2}(q+q^{\sigma_2}))}{\sqrt{1+p^2+q^2}\sqrt{1+p^{2\sigma_2}+q^{2\sigma_2}}}.$$

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